ROBUST FEEDBACK SYNTHESIS FOR A DISTURBED CANONICAL SYSTEM

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The statement of the feedback synthesis problem

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in Q \subset \mathbb{R}^n$, Q is a certain neighborhood of the origin; and $u \in \Omega \subset \mathbb{R}^r$, Ω is such that $0 \in \ int\, \Omega,\ f(0,0) = 0$.

Definition 1. *The local feedback synthesis* consists in constructing a control of the form $u = u(x),\ \ x \in Q$ such that:

1. $u(x) \in \Omega$;
2. the trajectory $x(t)$ of the closed-loop system

   $$\dot{x} = f(x, u(x)), \tag{2}$$

   starting at an arbitrary point $x(0) = x_0 \in Q$, ends at the origin $x(T) = 0$ at some finite time $T = T(x_0)$.

If $Q = \mathbb{R}^n$, this problem will be referred to as *the global feedback synthesis*.

The statement of the feedback synthesis problem

Remark 1. Let us note some difficulties related to solving this problem.

1. Since there exist infinitely many trajectories passing through the origin (note that the time of motion is finite), the right-hand side of equation (2) cannot satisfy the Lipschitz condition in a neighborhood of the origin according to the existence and uniqueness theorem.
2. The control $u(x)$ must satisfy the preassigned constraint $u(x) \in \Omega$.

Our approach is based on the *controllability function method* suggested by V. I. Korobov in 1979 as a way to solve the feedback synthesis problem.

Korobov V. I. A general approach to the solution of the bounded control synthesis problem in a controllability problem// Math. USSR Sb. – 1980. – 37(4). – pp. 535 - 557, translation from Mat. Sb.– 1979. –109(151). – No. 4(8).

The controllability function method

Consider the canonical system

$$\dot{x} = A_0 x + b_0 u, \tag{3}$$

where $x \in \mathbb{R}^n$, u satisfies the constraint $|u| \leq 1$,

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

Solving the synthesis problem for an arbitrary linear system with a scalar control can be reduced to solving the synthesis problem for the canonical system.

Korobov V. I. Controllability function method (Russian), M.-Izhevsk, R&C Dynamics. – 2007. – pp. 1-576.

The controllability function method

1. $$F^{-1} = \int_0^1 (1-t)e^{-A_0 t} b_0 b_0^* e^{-A_0^* t} dt = \qquad (5)$$
$$= \left(\frac{(-1)^{2n-i-j}}{(n-i)!(n-j)!(2n-i-j+1)(2n-i-j+2)} \right)_{i,j=1}^{n},$$
2. $$D(\Theta) = diag \left(\Theta^{-\frac{2n-2i+1}{2}} \right)_{i=1}^{n} \qquad (6)$$
3. Define the controllability function $\Theta = \Theta(x)$ as a unique positive solution of the equation
$$2a_0\Theta = (D(\Theta)FD(\Theta)x, x), \ x \neq 0, \quad \Theta(0) = 0, \qquad (7)$$
4. where the constant a_0 satisfies the inequality
$$0 < a_0 \leq \frac{2}{f_{nn}}. \qquad (8)$$

The controllability function method

Then the control

$$u(x) = -\frac{1}{2}\, b_0^* D(\Theta(x)) F D(\Theta(x)) x \qquad (9)$$

solves the global feedback synthesis for system (3) and satisfies the constraint $|u(x)| \leq 1$.
Moreover, in this case the equation $\dot{\Theta}(x) = -1$ holds, i.e., the controllability function equals the time of motion from any initial point $x_0 \in \mathbb{R}^n$ to the origin.

Korobov V. I. Controllability function method (Russian), M.-Izhevsk, R&C Dynamics. – 2007. – pp. 1-576.

The statement of the robust feedback synthesis problem

$$\begin{cases} \dot{x}_1 = (1 + p(t,x))x_2, \\ \dot{x}_2 = (1 + r_2 p(t,x))x_3, \\ \dots \\ \dot{x}_{n-1} = (1 + r_{n-1} p(t,x))x_n, \\ \dot{x}_n = u. \end{cases} \tag{10}$$

Here $t \geq 0$, $x \in \mathbb{R}^n$ is a state ($n \geq 2$), $u \in \mathbb{R}$ satisfies the constraint $|u| \leq 1$, r_i, $i = 2, \dots, n-1$ are given numbers, and $p(t,x)$ is an *unknown* perturbation, which, however, satisfies the constraint $d_1 \leq p(t,x) \leq d_2$. The case when p is a fixed parameter is well known.

Polyak B. T. and Shcherbakov P. S. Robust stability and control (Russian), Nauka, Moskva. – 2002. – pp. 1-303.

The statement of the robust feedback synthesis problem

We rewrite this system at the matrix form

$$\dot{x} = (A_0 + p(t,x)R)x + b_0 u, \tag{11}$$

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & r_2 & 0 & \dots & 0 & 0 \\ & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & r_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & r_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{12}$$

The robust feedback synthesis problem is considered for the case with a unique perturbation, i. e. in system (11) $r_i = 0$.

Korobov V. I., Revina T. V. Robust feedback synthesis problem for systems with a single perturbation // Communications in Mathematical Analysis. – 2014. – 17, no. 2. – pp. 217 – 230, www.math-res-pub.org/cma/17.

The statement of the robust feedback synthesis problem

Definition 2. By $\mathcal{P}_{d_1,d_2}$ we denote the set of functions $p(t,x) \; : [0;+\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

1. $p(t,x)$ is continuous in all variables;
2. in any domain $K_1(\rho_2) = \{(t,x) : \; 0 \le t < +\infty, \; \|x\| \le \rho_2\}$, the function $p(t,x)$ satisfies the Lipschitz condition $|p(t,x'') - p(t,x')| \le \ell_1(\rho_2)\|x'' - x'\|$, where $\ell_1(\rho_2)$ depends on the function p;
3. the function $p(t,x)$ satisfies the constraint $d_1 \le p(t,x) \le d_2$ for all $(t,x) \in [0;+\infty) \times \mathbb{R}$.

The statement of the robust feedback synthesis problem

Let $u(x)$ be the above-mentioned control which solves the synthesis problem for the canonical system

$$\dot{x} = A_0 x + b_0 u.$$

The goal of our work is

1. to find such a $d_1 < 0$ and a $d_2 > 0$ that the value of $d_2 - d_1$ becomes maximal for given numbers r_i,
2. for any perturbation $p(t, x) \in \mathcal{P}_{d_1, d_2}$ the trajectory $x(t)$ of the closed-loop system

$$\dot{x} = (A_0 + p(t, x)R)x + b_0 u(x), \tag{13}$$

   starting at an arbitrary initial point $x(0) = x_0 \in \mathbb{R}$, ends at the origin at some finite time $T(x_0, p(t, x))$.

This problem will be referred to as *the* (d_1, d_2) *global robust feedback synthesis (or robust finite-time stabilization).*

Main result

1. $$H = diag\left(-\frac{2n-2i+1}{2}\right)_{i=1}^{n}, \tag{14}$$

2. $$F^1 = F - FH - HF = ((2n-i-j+2)f_{ij})_{i,j=1}^{n}, \tag{15}$$

3. $$S_0 = FR + R^*F, \tag{16}$$

4. $$\dot{\Theta} = \frac{(-F^1 + p(t,x)S_0)y(\Theta,x), y(\Theta,x))}{(F^1 y(\Theta,x), y(\Theta,x))}, \tag{17}$$

where $y(\Theta,x) = D(\Theta)x$.

Main result

Theorem. Let us choose $0 < \gamma_1 < 1,\ \gamma_2 > 1$. Put

$$\tilde{d}_1^0 = 1/\lambda_{min}((F^1)^{-1}S_0); \quad \tilde{d}_2^0 = 1/\lambda_{max}((F^1)^{-1}S_0). \tag{18}$$

$$d_1^0 = \max\{(1-\gamma_1)\tilde{d}_1^0;\ (1-\gamma_2)\tilde{d}_2^0\}, \quad d_2^0 = \min\{(1-\gamma_1)\tilde{d}_2^0;\ (1-\gamma_2)\tilde{d}_1^0\}. \tag{19}$$

Then for all d_1 and d_2 such that $d_1^0 < d_1 < d_2 < d_2^0$, control (9) solves the (d_1, d_2) global robust feedback synthesis for system (10). The time of motion $T = T(x_0, d_1, d_2)$ is bounded as follows:

$$\frac{\Theta(x_0)}{\gamma_2} \le T(x_0, d_1, d_2) \le \frac{\Theta(x_0)}{\gamma_1}. \tag{20}$$

Remark 2. $d_2^0 - d_1^0$ is decreasing on γ_1 and increasing on γ_2.

To find a specific trajectory we act as follows. We take an arbitrary initial point $x_0 \in \mathbb{R}^n$. Then we solve equation (7) at $x = x_0$ and find its unique positive solution $\Theta(x_0) = \theta_0$. Put $\theta(t) = \Theta(x(t))$. For any perturbation $d_1^0 < d_1 \le p(t,x) \le d_2 < d_2^0$, the trajectory is a solution of the following Cauchy problem:

$$\begin{cases} \dot{x} = (A_0 + p(t,x)R)x - \frac{1}{2}\, b_0 b_0^* D(\theta) F D(\theta) x, \\ \dot{\theta} = \dfrac{((-F^1 + p(t,x)S_0)D(\theta)x, D(\theta)x)}{(F^1 D(\theta)x, D(\theta)x)} \\ x(0) = x_0,\ \theta(0) = \theta_0. \end{cases} \tag{21}$$

Notice that equation (7) is solved only once at initial point x_0 in order to determine θ_0.

An example: robust feedback synthesis problem for a two-dimensional system

Let us consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ u \end{pmatrix} + p(t, x_1, x_2) \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \tag{22}$$

under a constraint on the control of the form $|u| \leq 1$. In system (22), the function $p(t, x_1, x_2)$ is an *unknown* bounded perturbation which satisfies the constraint $d_1 \leq p(t, x_1, x_2) \leq d_2$.

An example: robust feedback synthesis problem for a two-dimensional system

This system can be written in the form:
$\dot{x} = (A_0 + p(t, x_1, x_2)R)x + b_0 u$, where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{23}$$

Since

$$rg(b_0, (A_0 + pR)b_0) = rg \begin{pmatrix} 0 & 1+p \\ 1 & 0 \end{pmatrix} = 2, \tag{24}$$

this system is completely controllable if $p(t, x_1, x_2) \equiv p \neq -1$.

An example: robust feedback synthesis problem for a two-dimensional system

Let us define the controllability function $\Theta = \Theta(x_1, x_2)$ as a unique solution of equation (7) which takes the form

$$2a_0\Theta^4 = 36x_1^2 + 24\Theta x_1 x_2 + 6\Theta^2 x_2^2, \tag{25}$$

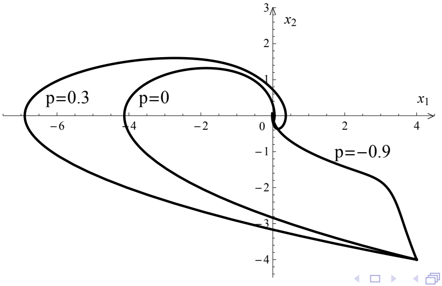
where $0 < a_0 \leq 2/f_{22} = 1/3$. Let $a_0 = 1/3$. Control (9) which solves the robust feedback synthesis problem is of the form

$$u(x_1, x_2) = -\frac{6x_1}{\Theta^2(x_1, x_2)} - \frac{3x_2}{\Theta(x_1, x_2)}. \tag{26}$$

This control satisfies the constraint $|u(x_1, x_2)| \leq 1$.

An example: robust feedback synthesis problem for a two-dimensional system

We get $-0.9 \leq p(t, x_1, x_2) \leq 0.3$.
Let the initial point be equal to $x_0 = (4; -4)$. Then the unique positive solution of equation (25) is $\Theta(x_0) = \theta_0 \approx 9,68$. The three trajectories corresponding to $p = -0,9$; $p = 0$ and $p = 0.3$ are given in Fig. 1. All the other trajectories fill up the domain between the trajectories corresponding to $p = -0.9$ and $p = 0.3$ at a perturbation which satisfy the inequality $-0.9 \leq p(t, x_1, x_2) \leq 0.3$.



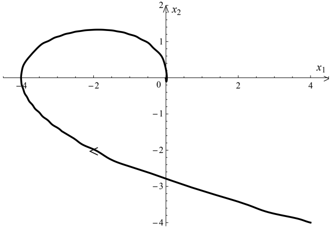
An example: robust feedback synthesis problem for a two-dimensional system

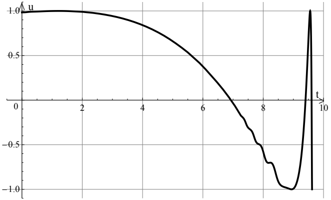
As a specific realization of the perturbation, let us consider the function $p = p(t, x_1, x_2) = -0.3 \sin((x_1^2 + x_2^2)t)$. The trajectory is a solution of the following Cauchy problem:

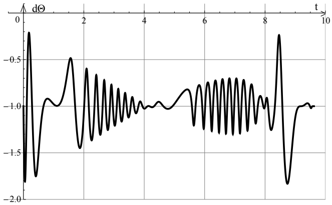
$$\begin{cases} \dot{x}_1 = (1 + p)\, x_2, \\ \dot{x}_2 = -\dfrac{6\, x_1}{\theta^2} - \dfrac{3x_2}{\theta}, \\ \dot{\theta} = \dfrac{-12\, x_1^2 + (-6 + 6p)\, x_1\, x_2\, \theta + (-1 + 2p)\, x_2^2\, \theta^2}{12x_1^2 + 6x_1\, x_2\, \theta + x_2^2\, \theta^2}, \\ x_1(0) = 0.12, \quad x_2(0) = -0.1, \quad \theta(0) = 9,68. \end{cases} \tag{27}$$

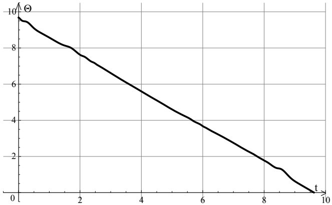
It can be shown numerically that the time of motion is approximately equal to $T \approx 9.62$. Notice that the time of motion is less than θ_0. The estimate of the time of motion (20) gives a rough result: $2.42 \leq T \leq 107.56$.

An example: robust feedback synthesis problem for a two-dimensional system









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