One special construction in the spectral theory of C_0-semigroups

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Grigory M. Sklyar, Vitalii Marchenko, Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors, http://arxiv.org/abs/1405.2731

Abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \geq 0, \\ x(0) = x_0 \in H, \end{cases} \tag{1}$$

Operator A generates a C_0-semigroup on H if and only if Cauchy problem (1) is well-posed and $\rho(A) \neq \emptyset$.

For example, this phenomenon takes place for:

1. Maxwell's equations of electrodynamics
2. Neutral type systems of delay differential equations
3. Regular Sturm-Liouville systems
4. Linear heat, diffusion and wave equations

Theorem (H. Zwart, J. Differential Equations, 2010; G.Q. Xu & S.P. Yung, J. Differential Equations, 2005)

Let A be the generator of the C_0-group on H with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ (counting with multiplicity) and the corresponding (normalized) eigenvectors $\{e_n\}_{n=1}^{\infty}$. If the following two conditions hold,

1. $\overline{\mathrm{Lin}}\{e_n\}_{n=1}^{\infty} = H$;
2. The point spectrum has a uniform gap, i.e.,

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0, \tag{2}$$

then $\{e_n\}_{n=1}^{\infty}$ forms a Riesz basis of H.

The main purpose

To show that the assumption (2) is essential.

Different cases

- The case when $\{\lambda_n\}_{n=1}^{\infty}$ can be decomposed into $K < \infty$ sets, with every set satisfying (2), was considered by H. Zwart. More precisely, Let the eigenvalues of the generator A of the C_0-group on H can be grouped into K sets, i.e., $\{\lambda_n\}_{n=1}^{\infty} = \bigcup_{j=1}^{K} \{\lambda_{n,j}\}_{n=1}^{\infty}$, with $\inf_{n \neq m} |\lambda_{n,k} - \lambda_{m,k}| > 0$, $k = 1, \ldots, K$, and the span of the generalized eigenvectors of A is dense. Then there exists a sequence of spectral projections $\{P_n\}_{n=1}^{\infty}$ of A such that $\{P_n H\}_{n=1}^{\infty}$ forms a Riesz basis of subspaces in H with $\max_n \dim P_n H = K$.
- So we consider the case when $\{\lambda_n\}_{n=1}^{\infty}$ does not satisfy (2) and cannot be decomposed into $K < \infty$ sets, with every set satisfying (2).

Space $H_1^0(\{e_n\})$

We start with a separable Hilbert space H. Consider arbitrary Riesz basis $\{e_n\}_{n=1}^{\infty}$ of H, operator $T : e_n \mapsto e_{n+1}$, $n \in \mathbb{N}$, and introduce the space

$$H_1^0(\{e_n\}) = \{x \in H : \ \|x\|_1 = \|(I - T)x\|\}.$$

$H_1^0(\{e_n\})$ is a normed linear space, but incomplete, since $0 \in \sigma(I - T)$.

Space $H_1(\{e_n\})$

By $H_1(\{e_n\})$ we denote the completion of $H_1^0(\{e_n\})$ in the norm $\|\cdot\|_1$.

It turns out that

the space $H_1(\{e_n\})$ consists of formal series $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n$ with the property

$$\{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_2.$$

And $H_1(\{e_n\})$ is a Hilbert space with a norm

$$\|x\|_1 = \left\| (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum_{n=1}^{\infty} (c_n - c_{n-1}) e_n \right\|,$$

$x \in H_1(\{e_n\})$, and a scalar product
$\langle x, y \rangle_1 = \langle (I - T) x, (I - T) y \rangle, \ x, y \in H_1(\{e_n\})$.

Example

Given any $\alpha \in [0, \frac{1}{2})$, we have $(\mathfrak{f}) \sum_{n=1}^{\infty} n^{\alpha} e_n \in H_1(\{e_n\})$.

Remark

In particular case of $H = \ell_2$ and when $\{e_n\}_{n=1}^{\infty}$ is the canonical basis of ℓ_2, $H_1(\{e_n\}) = \ell_2(\Delta)$, where $\ell_2(\Delta) = \{x = \{\alpha_n\}_{n=1}^{\infty} : \Delta x \in \ell_2\}$, Δ is a difference operator,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The space $\ell_2(\Delta)$ was first introduced by F. Başar and B. Altay in 2003. (Ukrainian Math. J.)

So we have that

$$H_1(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\}$$

Proposition (Some properties of the space $H_1(\{e_n\})$)

1. $\overline{\mathrm{Lin}}\{e_n\}_{n=1}^{\infty} = H_1(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $H_1(\{e_n\})$;
3. $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system
$$\left\{\chi_n = (I-T)^{-1}(I-T^*)^{-1}e_n^*\right\}_{n=1}^{\infty}$$
in $H_1(\{e_n\})$, where $\langle e_n, e_m^*\rangle = \delta_n^m$;
4. $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $H_1(\{e_n\})$ while the sequence $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal in $H_1(\{e_n\})$;
5. $H_1(\{e_n\})$ is a separable Hilbert space, isomorphic to ℓ_2;
6. $H_1(\{e_n\})$ has an orthonormal basis.

The operator $A : H_1(\{e_n\}) \supset D(A) \mapsto H_1(\{e_n\})$

$$Ax = A(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} \lambda_n c_n e_n, \qquad (3)$$

where $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$, and domain

$$D(A) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \ \{\lambda_n c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\}. \qquad (4)$$

Theorem

Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis of H. Then $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $H_1(\{e_n\})$ and the operator A defined by (3) with domain (4), where $\lambda_n = i \ln n$, generates a C_0-group on $H_1(\{e_n\})$.

The crucial step of the proof

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2 \leq 4\sum_{n=1}^{\infty}a_n^2 \quad \text{– Discrete Hardy inequality.}$$

$$\left\|\sum_{n=2}^{\infty}e^{it\ln n}c_{n-1}\frac{it}{n}e_n\right\|^2 = |t|^2\left\|\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1}(c_j-c_{j-1})\frac{e^{it\ln n}}{n}\right)e_n\right\|^2 \leq$$

$$M|t|^2\sum_{n=2}^{\infty}\left|\sum_{j=1}^{n-1}\frac{c_j-c_{j-1}}{n}\right|^2 \leq M|t|^2\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n}|c_j-c_{j-1}|\right)^2 \leq$$

$$4M|t|^2\sum_{n=1}^{\infty}|c_n-c_{n-1}|^2 \leq 4\frac{M}{m}|t|^2\left\|\sum_{n=1}^{\infty}(c_n-c_{n-1})e_n\right\|^2 = 4\frac{M}{m}|t|^2\|x\|_1^2.$$

Definition

Let $f : [1, +\infty) \mapsto \mathbb{R}$. Then we define the following function class,

$$\mathcal{S} = \left\{ f : \lim_{x \to \infty} f(x) = +\infty; \lim_{n \to \infty} n|f(n-1) - f(n)| < \infty \right\}.$$

For example,

$f(x) = \ln x \in \mathcal{S}$, $v(x) = \ln \ln(x+1) \in \mathcal{S}$, $g(x) = \ln \ln \sqrt{x+1} \in \mathcal{S}$.

Theorem (Generalization)

Assume that $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H. Then $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $H_1(\{e_n\})$ and the operator $A : H_1(\{e_n\}) \supset D(A) \mapsto H_1(\{e_n\})$, defined by

$$Ax = A(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} if(n) \cdot c_n e_n,$$

where $f \in \mathcal{S}$, with domain

$$D(A) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \ \{f(n) \cdot c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\},$$

generates a C_0-group on $H_1(\{e_n\})$.

Surprisingly!

Even if we consider the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of A defined by (3,4) of the same geometric nature, i.e. satisfying

$$\lim_{n\to\infty} i\lambda_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0, \tag{5}$$

then A not necessary generates a C_0-group on $H_1(\{e_n\})$.

Proposition (Negative result)

Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis in H. Then the operator A defined by (3) with domain (4), where $\{\lambda_n\}_{n=1}^{\infty}$ satisfy (5) and $\lim_{n\to\infty} \frac{|\lambda_n|}{\sqrt{n}} > 0$, does not generate a C_0-group on $H_1(\{e_n\})$.

For instance, we can take $\lambda_n = i\sqrt{n}$.

The underlying reason of the phenomenon above is as follows.

The rotations

$$e^{it\sqrt{x}},$$

are slowing down, when $x \to +\infty$, with too low speed to guarantee the convergence of the series

$$\sum_{n=2}^{\infty} e^{it\sqrt{n}} c_{n-1} \frac{e_n}{\sqrt{n}}.$$

in the original space H.

The question is: What's going on between $i \ln n$ and $i\sqrt{n}$?

Proposition

Constructed C_0-group $\{e^{At}\}_{t\in\mathbb{R}}$ has the following remarkable properties.

1. The growth bound ω_0 of the C_0-group $\{e^{At}\}_{t\in\mathbb{R}}$ coincides with the spectral bound of its generator A and equals to 0.
2. The C_0-group $\{e^{At}\}_{t\in\mathbb{R}}$ grows as $|t|$ when t tends to $\pm\infty$.

The answer to the following question does not yet exist

Is it possible to construct the unbounded generator of a C_0-group with bounded non-Riesz basis family of eigenvectors?
Our conjecture is: Yes.
How to construct it?

Remark

The construction of unbounded generator of a C_0-semigroup with bounded non-Riesz basis family of eigenvectors is quite simple. (M. Haase, The functional calculus for sectorial operators, 2006)

Thanks for the attention!