Analogue of topological entropy for some infinite-dimensional systems

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Let (X, d) be metric space, $T \in C(X \to X)$. Metric d_n on X is defined as

$$d_n(x, y) = \max_{0 \leq j \leq n} d(T^j(x), T^j(y)).$$

In other words, it is the maximum distance between the orbits of x and y after n iterations. As we know, the set is called (n, ε)-separated for fixed $\varepsilon > 0$ if pairwise distances between its points in metric d_n are not less then ε. The cardinality of the biggest of such sets is denoted as $N(n, \varepsilon)$.

Then the topological entropy of function T is defined by

$$h(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\ln N(n, \varepsilon)}{n}.$$

However the definition of the topological entropy is not effective for describing the complexity of infinite-dimensional dynamical systems. In particular, for dynamical systems generated by difference equations with continuous argument $x(t+1) = f(x(t))$, ie when $X = C([0,1], I)$ and T has the form $T(\varphi)(t) = f(\varphi(t)),\ f \in C(I, I)$, the value of topological entropy is either 0 or $+\infty$. Thus such definition of entropy does not help us to conduct thorough analysis of such systems. The purpose – to offer an analogue of definition of entropy, by which it is possible to evaluate the complexity of such dynamical systems more effectively.

For systems considered above we gain the analogue, which can take finite values, by multiplying the topological entropy by ε :

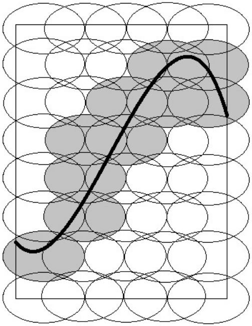
$$\tilde{h}(T) = \limsup_{\varepsilon \to 0} \varepsilon \lim_{n \to \infty} \frac{\ln N(n, \varepsilon)}{n}.$$

This introduced value $\tilde{h}(T)$ is finite and nonzero iff the entropy of one-dimensional function f is finite and nonzero.

The idea of the proof is based on estimating the value $N(n, \varepsilon)$.
If $M(n, \varepsilon)$ – maximum (n, ε)-separated set for f, we can show that in Hausdorff metric:

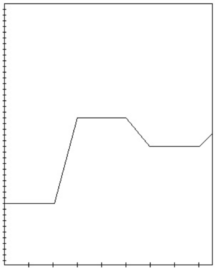
$$N(n, \varepsilon) \leq (M(n, \varepsilon/4)^2/2)^{([2/\varepsilon]+1)}.$$

By using this inequality it is easy to get $\tilde{h}(T) \leq 4h(f)$, which proves finiteness $\tilde{h}(T)$ in the case of finite $h(f)$.



The inequality
$N(n, \varepsilon) \leq (M(n, \varepsilon/4)^2/2)^{[2/\varepsilon]+1}$
follows from the next reasons.
We can construct such cover of $[0,1] \times I$, that the distance d_n between two functions intersecting the same set of elements of this cover, is less than ε.
Moreover this cover consists $[2/\varepsilon] + 1$ columns, each with $M(n, \varepsilon/4)$ elements.
We can establish a correspondence between the continuous function and a set of elements that this function intersects. By estimating the number of such sets, we get the inequality.

In the other side the value of $N(n, \varepsilon)$ can be estimated by constructing $M(n, \varepsilon)^{[\frac{1}{3\varepsilon}]}$ (n, ε)-separated functions. These functions are defined as horizontal segments of length 2ε at different heights, joined by segments with the base of ε. The height of each of horizontal segments will be one of the values of some maximum (n, ε)-separated set. We can build $M(n, \varepsilon)^{[\frac{1}{3\varepsilon}]}$ such functions.



The gained functions will be $(n, \varepsilon)-$separated for T, so $N(n, \varepsilon) \geq M(n, \varepsilon)^{[\frac{1}{3\varepsilon}]}$. Thus $\tilde{h}(T) \geq \frac{1}{3} h(f)$.

In the more general case when $X = C(K, I)$ and T has the form $T(\varphi)(t) = f(\varphi(t)),\ f \in C(I, I)$, where K – any compact, we can consider value

$$\tilde{h}(T) = \limsup_{\varepsilon \to 0}\ N_K(\varepsilon)^{-1} \lim_{n \to \infty} \frac{\ln N(n, \varepsilon)}{n},$$

where $N_K(\varepsilon)$ denotes the minimum number of elements in the ε-cover of compact K by open balls.
Through similar reasoning, we can prove that this value is finite when the $h(f)$ is finite.
In addition, if $K \subset R^d$, then $\tilde{h}(T) > 0$, where $h(f) > 0$.

If we consider the system of $(C(K, L), T)$, where K, L – arbitrary compacts, then for the evaluation of its complexity we need to analyze the properties of the compacts K and L, which are related to the number of elements in their $\varepsilon-$covers and with the structure of this $\varepsilon-$covers, such as mean dimension.

Thank you for your attention!