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ON DYNAMICAL SYSTEM OF CONFLICT WITH FAIR REDISTRIBUTION OF VITAL RESOURCES

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We discuss the properties of dynamical system of conflict (DSC). The notion of DSC was introduced in works [6, 7]) for modelling the alternative interaction between opponents.

Let the probability measures μ, ν describe the starting distribution of the vital resource space Ω for a pair of opponents A, B.

The problem is to find the **law** of conflict interaction $\star$ between A, B which ensures the compromise redistribution of Ω.

Assume that the evolution changes of $\mu(t), \nu(t)$ are governed by the following nonlinear law of conflict dynamic (cf. with [1, 5]):

$$\dot{\mu} = \frac{\mu\Theta - \tau}{z}, \qquad \dot{\nu} = \frac{\nu\Theta - \tau}{z},$$

where $\Theta = \Theta(\mu, \nu)$ is a positive quadratic form which fixes the so called conflict exponent describing the

global conflict interaction

between opponents A, B and $\tau = \tau(\mu, \nu)$ has sense of the

occupation

exponent (the local interaction).

The meanings of τ at each moment of time show the values of presence of opponents A, B on the opposite territory. The denominator z ensures that measures $\mu(t), \nu(t)$ are probability for all $t > 0$.

We prove that under the appropriate construction of Θ and τ there exists the ω-limit state $\{\mu^\infty, \nu^\infty\}$ which corresponds to the

fair redistribution

of the vital resources space Ω between opponents A, B. The fair redistribution means that the opponents A, B reach the

compromise, equilibrium state.

That is the limit measures μ^∞, ν^∞ coincide with the normalized components of the classic Jordan decomposition of the signed measure $\omega = \mu - \nu = \omega_+ - \omega_-$, i.e.

$$\mu^\infty = \mu_+ := \frac{\omega_+}{\omega_+(\Omega)}, \quad \nu^\infty = \nu_- := \frac{\omega_-}{\omega_-(\Omega)}.$$

In what follows we come to the discrete time $t = N = 0, 1, ...$ and use the system of difference equations:

$$\begin{cases} \mu^{N+1}(E) = \mu^N(E) + \mu^N(E)\Theta^N - \tau^N(E), \\ \nu^{N+1}(E) = \nu^N(E) + \nu^N(E)\Theta^N - \tau^N(E), \ E \in \mathcal{B}, \end{cases} \tag{1}$$

where we omit normalization denominators.

In [11] it was proved (see also [6, 7, 12]) that each trajectory $\{\mu^N, \nu^N\}, N \geq 1$ starting with any couple of probability measures $\mu, \nu \in \mathcal{M}_1^+(\Omega),\ \mu \neq \nu$, converges in the weak sense to a limit fixed point $\{\mu^\infty, \nu^\infty\}$. This point creates an equilibrium state for the system and is compromise in the sense $\mu^\infty \perp \nu^\infty$. Moreover, for each dynamical system of conflict $\{\Omega, \mathcal{M}_1^+(\Omega), \star\}$ given by (1), there exists the limit ω-set Γ^∞ [13]. It is attractor consisting all couples of mutually singular measures from $\mathcal{M}_1^+(\Omega)$. Thus,

$$\Gamma^\infty = \{\{\mu^\infty, \nu^\infty\} \mid \mu^\infty, \nu^\infty \in \mathcal{M}_1^+(\Omega),\ \mu^\infty \perp \nu^\infty\}.$$

In the simplest situation the dynamical system of conflict can be written in terms of coordinates of stochastic vectors $p, r \in \mathbf{R}^n_+, n \geq 2$ corresponding to the opponent sides:

$$p_i^{N+1} = 1/z^N(p_i^N \Theta^N - \tau_i^N), \quad r_i^{N+1} = 1/z^N(r_i^N \Theta^N - \tau_i^N), \; i = 1, ..., n.$$

Here $\Theta^N = (p^N, r^N)$ be the inner product between vectors p^N, r^N and $\tau_i^N = p_i^N r_i^N$. We claim that each trajectory $\{p^N, r^N\}_{N=0}^{\infty}$ starting with a couple of stochastic vectors $\{p^0 = p, r^0 = r\}, \; p \neq r$ converges with $N \to \infty$ to a fixed point $\{p^\infty, r^\infty\}$ which creates a compromise state, $p^\infty \perp r^\infty$. This state is uniquely determined by the starting couple $\{p, r\}$ and has an explicit coordinate representation:

$$p_i^\infty = \frac{d_i}{D} > 0, \; i \in N_+, \;\; r_k^\infty = -\frac{d_k}{D} > 0, \; k \in N_-, \;\; D = 1/2 \sum_{i=1}^{n} |d_i|, \tag{2}$$

$$p_i^\infty = 0, \; i \notin N_+, \; r_k^\infty = 0, k \notin N_-,$$

where

$$d_i = p_i - r_i, \quad N_+ = \{i : d_i > 0\}, \quad N_- = \{k : d_k < 0\}.$$

In [10, 9] we generalized above constructions for the cases of piecewise uniformly distributed measures, self-similar, and similar structure measures.

In the most general situation we obtain the similar results in terms of abstract measures (for detail see [11, 12]).

We will deal with DSC $\{\Omega, \mathcal{M}^+_{1,\mathrm{ac}}(\Omega), \star\}$ of natural conflict (here $\mathcal{M}^+_{1,\mathrm{ac}}$ denotes a class of the absolutely continuous measures). A term "natural" means that conflict composition $\star$ is defined by a fixed law of the conflict interaction between opponents and their strategies do not change during the time evolution.

Let us consider an abstract variant of DSC at discrete time

$$\mu^{N+1} = \mu^N \star \nu^N, \ \nu^{N+1} = \nu^N \star \mu^N, \ N = 0, 1...$$

Their state trajectories

$$\left\{ \begin{array}{c} \mu^N \\ \nu^N \end{array} \right\} \xrightarrow{\star} \left\{ \begin{array}{c} \mu^{N+1} \\ \nu^{N+1} \end{array} \right\}, \quad N = 0, 1, ... \tag{3}$$

are governed by the following *law of conflict dynamic*:

$$\left\{ \begin{array}{l} \mu^{N+1}(E) = \frac{1}{z^N}[\mu^N(E)(\Theta^N + 1) - \tau^N(E)], \\ \nu^{N+1}(E) = \frac{1}{z^N}[\nu^N(E)(\Theta^N + 1) - \tau^N(E)], \quad E \in \mathcal{B}, \end{array} \right. \tag{4}$$

where measures $\mu^0 = \mu, \ \nu^0 = \nu$ correspond to an initial state. The conflict exponent Θ^N in (4) is defined as

$$\Theta^N = \int_\Omega \int_\Omega \mathcal{K}(x, y) \varphi^N(x) \psi^N(y) dx dy,$$

where $\mathcal{K}(x, y)$ denotes a kernel of some positive bounded operator K in $L^2(\Omega, \ d\lambda)$ and

$$\varphi^N(x) = \sqrt{\rho^N(x)}, \ \psi^N(x) = \sqrt{\sigma^N(x)},$$

where $\rho^N(x), \sigma^N(x)$ are the Radon-Nikodym derivatives of μ^N, ν^N with respect to λ. Thus,

$$\Theta_N = (K\varphi^N, \psi^N)_{L^2(\Omega, \ d\lambda)},$$

Further, τ^N in (4) stands for the occupation measure. Its values characterize the presence of opponents on opposite territories. By definition,

$$\tau^N(E) = \nu^N(E_+) + \mu^N(E_-), \; E_+ = E \bigcap \Omega_+, \; E_- = E \bigcap \Omega_-, \quad (5)$$

where $\Omega = \Omega_- \cup \Omega_+$ corresponds to the Hahn-Jordan decomposition (see [3, 14]) of the starting signed measure $\omega = \mu - \nu$. Finally, the normalizing denominator in (4) is defined as

$$z^N = \Theta^N + 1 - W^N, \quad W^N = \mu^N(\Omega_-) + \nu^N(\Omega_+).$$

It is easy to see that all measures $\mu^N, \nu^N, \quad N \geq 1$ in (4) are absolutely continuous and probability, i.e., $\mu^N, \nu^N \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$.

The DSC defined by (4) has two separate sets of fixed points. The first set contains all couples of identical measures from $\mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$. Indeed, if $\mu = \nu$, then Θ^N =const for all N and $\mu^N(E) = \mu(E) = \nu^N(E) = \nu(E)$ for each $E \in \mathcal{B}$. The second set is composed from measures $\mu, \nu \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$ which are orthogonal, $\mu \perp \nu$. In this case $\tau^N = 0 = W^N$ and $\Theta^N + 1 = z^N$ for all N. Due to (4) we find that $\mu^N = \mu, \; \nu^N = \nu$.

In all other cases, when the starting measures are different, $\mu \neq \nu$, and mutually non-singular the following theorem is true.

Theorem Let $\{\Omega, \mathcal{M}^+_{1,\mathrm{ac}}(\Omega), \star\}$ be a DSC generated by the system of difference equations (4). Then each its trajectory (3) starting with a couple of probability measures $\mu^0 = \mu, \nu^0 = \nu \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$, $\mu \neq \nu$ converges to the fixed point corresponding to the limit state $\{\mu^\infty, \nu^\infty\}$ with

$$\mu^\infty(E) = \lim_{N\to\infty} \mu^N(E), \;\; \nu^\infty(E) = \lim_{N\to\infty} \nu^N(E), \;\; E \in \mathcal{B}, \qquad (6)$$

where

$$\mu^\infty(E) = \frac{\mu(E_+) - \nu(E_+)}{D} = \mu_+(E), \qquad (7)$$

$$\nu^\infty(E) = -\frac{\mu(E_-) - \nu(E_-)}{D} = \nu_-(E)$$

with μ_+, ν_- defined as the Hahn normalized components of $\omega = \mu - \nu$.

In (7) $D = 1/2 \int_\Omega |\rho(x) - \sigma(x)| dx$ stands for the total difference between measures μ, ν.

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