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Infnite orbit equivalence class for a minimal substitution dynamical system

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Cantor dynamical systems

A Cantor set X is a 0-dimensional compact metric space without isolated points.

A Cantor system is a pair (X, T) where X is a Cantor set and $T: X \to X$ is a homeomorphism.

A Cantor system (X, T) is minimal if for every $x \in X$ the set $Orb_T(x) = \{T^n(x) \mid n \in \mathbb{Z}\}$ is dense in X.

A Cantor system is called uniquely ergodic if it has a unique invariant probability measure.

Two Cantor systems (X, T) and (Y, S) are isomorphic if there exists a homeomorphism $h: X \to Y$ such that $h \circ T = S \circ h$.

Cantor dynamical systems

Two Cantor systems (X, T) and (Y, S) are called orbit equivalent if there exists a homeomorphism $h\colon X \to Y$ such that $h(Orb_T(x)) = Orb_S(h(x))$ for every $x \in X$.

Let μ be a Borel probability measure on a Cantor set X. The clopen values set $S(\mu) = \{\mu(U) : U \text{ clopen in } X\}$.

For a non-atomic probability measure μ, the set $S(\mu)$ is a countable dense subset of the unit interval.

Theorem (Giordano, Putnam, Skau (1995))

Two minimal uniquely ergodic Cantor systems (X, T, μ) and (Y, S, ν) are orbit equivalent if and only if $S(\mu) = S(\nu)$.

Substitution dynamical systems

Let $\mathcal{A} = \{a_1, ..., a_s\}$ be a finite alphabet.
Let $\mathcal{A}^*$ be a collection of finite non-empty words over $\mathcal{A}$.

A substitution σ is a map $\sigma \colon \mathcal{A} \to \mathcal{A}^*$.

Let $A_\sigma = (a_{ij})_{i,j=1}^{s}$ be the incidence matrix associated to σ where a_{ij} is the number of occurrences of a_i in $\sigma(a_j)$.

$$\sigma = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

$$A_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Substitution dynamical systems

Substitution σ extends to maps $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ and $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by concatenation.

For $x \in \mathcal{A}^{\mathbb{Z}}$, let $L_n(x)$ be the set of all words of length n occurring in x. Set $L(x) = \bigcup_{n \in \mathbb{N}} L_n(x)$.

The language of σ is the set L_σ of all finite words occurring in $\sigma^n(a)$ for some $n \geq 0,\ a \in \mathcal{A}$. Set $X_\sigma = \{x \in \mathcal{A}^{\mathbb{Z}} : L(x) \subset L_\sigma\}$. The set X_σ is either finite or a Cantor set (in the induced topology).

Denote by T the shift on $\mathcal{A}^{\mathbb{Z}}$: $T(\ldots x_{-1}.x_0 x_1 \ldots) = \ldots x_{-1} x_0 . x_1 \ldots$.

The dynamical system (X_σ, T_σ), where T_σ is the restriction of T to the T-invariant set X_σ, is called the substitution dynamical system associated to σ.

A substitution σ is called primitive if there is n such that for each $a_i, a_j \in \mathcal{A}$, a_j appears in $\sigma^n(a_i)$.

Theorem (Michel (1974), Dekking (1978))

Every primitive substitution generates a minimal and uniquely ergodic dynamical system.

The complexity of $u \in \mathcal{A}^{\mathbb{Z}}$ is the function $p_u : \mathbb{N} \to \mathbb{N}$ which associates to each integer $n \geq 1$ the cardinality of $L_n(u)$.

For a periodic sequence $u =$ ...01.0101..., $p_u(n) = 2$ for every n.

For the sequence $v =$ ...011011100101... made by concatenating the binary expansions of integers 0,1,2,... we have $p_v(n) = 2^n$ for every n.

Let σ be a primitive substitution. Then for any $x, y \in X_\sigma$ and for every $n \in \mathbb{N}$: $p_x(n) = p_y(n)$. Denote by p_σ the function p_x for some $x \in X_\sigma$.

Theorem (Cobham (1972), Pansiot (1984), Ferenczi (1999))

Let σ, ζ be primitive substitutions.

(1) *There exists a constant $C_\sigma > 0$ such that $p_\sigma(n) \leq C_\sigma n$ for every $n \geq 1$.*

(2) *If the substitution dynamical systems (X_σ, T_σ) and (X_ζ, T_ζ) are isomorphic, then there exists a constant c such that, for all $n > c$,*

$$p_\sigma(n-c) \leq p_\zeta(n) \leq p_\sigma(n+c).$$

Hence a relation $p_\sigma(n) \leq an^k + \bar{o}(n^k)$ when $n \to \infty$ is preserved by isomorphism.

A substitution σ on an alphabet $\mathcal{A}$ is called proper if there exists an integer $n > 0$ and two letters $a, b \in \mathcal{A}$ such that for every $c \in \mathcal{A}$, a is the first letter and b is the last letter of $\sigma^n(c)$.

Theorem (Forrest (1997), Durand, Host, Skau (1999))

The class of simple properly ordered stationary Bratteli diagrams describes exactly primitive proper substitution dynamical systems.

Let μ be an invariant probability measure for a proper primitive substitution dynamical system (X_σ, T_σ). Let A_σ be the incidence matrix corresponding to σ. Let λ be the Perron-Frobenius eigenvalue of A_σ and $x = (x_1, ..., x_s)^T$ be the corresponding eigenvector. Let H be the additive subgroup of $\mathbb{R}$ generated by $\{x_1, \ldots, x_s\}$. Then the clopen values set

$$S(\mu) = \left(\bigcup_{N=0}^{\infty} \frac{1}{\lambda^N} H \right) \cap [0, 1].$$

Denote $S(A_\sigma) = S(\mu)$ **(Bezuglyi-K. (2011), (Bezuglyi-Kwiatkowski-Medynets (2009)).**

Main results

Theorem

Let σ be a proper primitive substitution. Then there exist countably many proper primitive substitutions $\{\zeta_n\}_{n=1}^{\infty}$ such that (X_σ, T_σ) is orbit equivalent to $(X_{\zeta_n}, T_{\zeta_n})$, but the systems $\{(X_{\zeta_n}, T_{\zeta_n})\}_{n=1}^{\infty}$ are pairwise non-isomorphic.

Ideas of the proof

(1) Increase the number of letters in the alphabet.

Lemma

Let $A \in Mat(\mathbb{N}, s)$ be a primitive matrix. Then there exist primitive matrices $\{A_n\}_{n=1}^{\infty}$, where $A_n \in Mat(\mathbb{N}, s+n)$ such that $S(A) = S(A_n)$ for all $n \in \mathbb{N}$.

Main results

Sketch of proof (1)

On each step i we built ζ_i such that ζ_i is orbit equivalent to σ and $\{\zeta_1, ..., \zeta_i\}$ are pairwise non-isomorphic.

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- Set $\zeta_1(a_j) = a_1 a_j \ldots a_1$, $j = 1, \ldots, C+2$
- (X_σ, T_σ) is orbit equivalent to $(X_{\zeta_1}, T_{\zeta_1})$
- $p_{\zeta_1}(n) \geq (C+1)n$, $n \geq 1 \Longrightarrow (X_\sigma, T_\sigma)$ is not isomorphic to $(X_{\zeta_1}, T_{\zeta_1})$

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- $p_{\zeta_1}(n) \geq (C+1)n$, $n \geq 1$:

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  $p_u(1) = C + 2$;

  $p_u(n+1) - p_u(n) \geq C + 1$ for $n \geq 1$

- There exists C_1 such that $p_{\zeta_1}(n) < C_1 n$, $n \geq 1$

  $(C+1)n \leq p_{\zeta_1}(n) < C_1 n$

Main results

(2) Use the powers $\{\sigma^n\} \Rightarrow$ increase the length of words in substitution, permute the letters (new incidence matrices are the powers of A_σ) $\Rightarrow$ the complexity function increases, orbit equivalence is preserved.

Theorem (Cassaigne (1997))

Let σ be a primitive substitution. Let $p_\sigma(n) \leq Cn$ for all $n \geq n_0$ and $C > 0$. Then for all $n \geq n_0$

$$p_\sigma(n+1) - p_\sigma(n) \leq KsC^3,$$

where s is the number of letters in the alphabet and K is a constant which does not depend on σ.

Main results

Theorem

Let σ be a primitive substitution whose incidence matrix has a Perron-Frobenius eigenvalue λ and $k = \deg \lambda$. Then (X_σ, T_σ) is orbit equivalent to a Bratteli-Vershik system defined on a stationary Bratteli diagram with k vertices on each level. Moreover, there is no stationary Bratteli-Vershik system with less than k vertices which is orbit equivalent to (X_σ, T_σ).

Idea: $S(A_\sigma) \subset \mathbb{Q}(\lambda) = \mathbb{Q}[\lambda] \longleftrightarrow \mathbb{Q}^k \subset \mathbb{R}^k$

S. Bezuglyi and O. Karpel, *Orbit Equivalent Substitution Dynamical Systems and Complexity*, Proc. Amer. Math. Soc. **142** (2014), 4155-4169.

Thank you for your attention!