# LIMIT CYCLE BIFURCATIONS

Valery A. Gaiko

United Institute of Informatics Problems
National Academy of Sciences of Belarus
Minsk, Belarus

Kluwer Academic Publishers

# Global Bifurcation Theory and Hilbert's Sixteenth Problem

Valery A. Gaiko

Book Series:
MATHEMATICS AND ITS APPLICATIONS
**Volume 562**

This volume is devoted to the qualitative investigation of two-dimensional polynomial dynamical systems and is aimed at solving Hilbert's Sixteenth Problem on the maximum number and relative position of limit cycles. The author presents a global bifurcation theory of such systems and suggests a new global approach to the study of limit cycle bifurcations.

The obtained results can be applied to higher-dimensional dynamical systems and can be used for the global qualitative analysis of various mathematical models in mechanics, radioelectronics, in ecology and medicine.

**Audience:** The book would be of interest to specialists in the field of qualitative theory of differential equations and bifurcation theory of dynamical systems. It would also be useful to senior level undergraduate students, postgraduate students, and specialists working in related fields of mathematics and applications.

**Kluwer Academic Publishers, Boston**
Hardbound, ISBN 1-4020-7571-5, August 2003, 204 pp.
http://www.wkap.nl/prod/b/1-4020-7571-5
EUR 112.00 / USD 125.00 / GBP 78.00

# Hilbert's Sixteenth Problem

**Problem.** *To find the maximum number and to determine the relative position of limit cycles of the equation*

$$\frac{dy}{dx} = \frac{Q_n(x,y)}{P_n(x,y)} \qquad (*)$$

*or of the corresponding dynamical system*

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \qquad (**)$$

*where* $P_n$ *and* $Q_n$ *are polynomials with real coefficients in real variables* $x$, $y$ *and not greater than* $n$ *degree.*

## Principal Bifurcations of Limit Cycles

- **Andronov–Hopf bifurcation**
  from a singular point of center or focus type
  (Fig. 1)

- **Separatrix cycle bifurcation**
  from a singular closed trajectory
  (Fig. 2)

- **Multiple limit cycle bifurcation**
  from a multiple limit cycle
  (Fig. 3)

# Principal Bifurcations

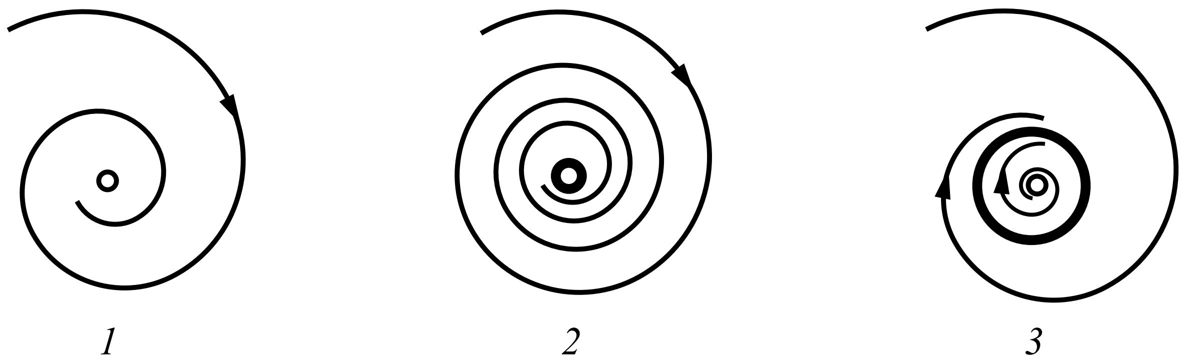


**Figure 1.** Andronov–Hopf bifurcation

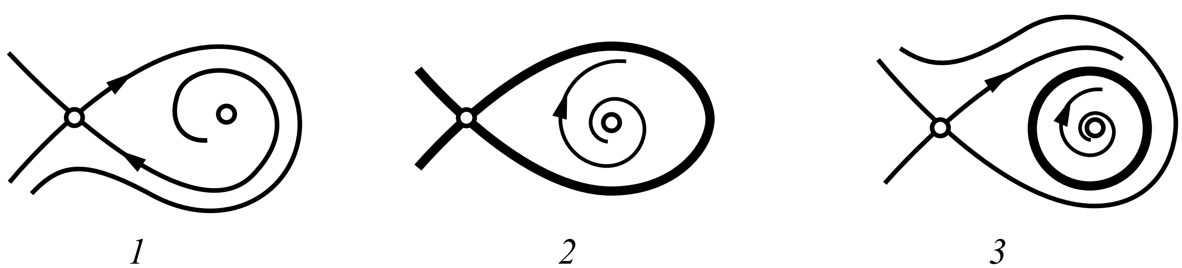


**Figure 2.** Separatrix cycle bifurcation

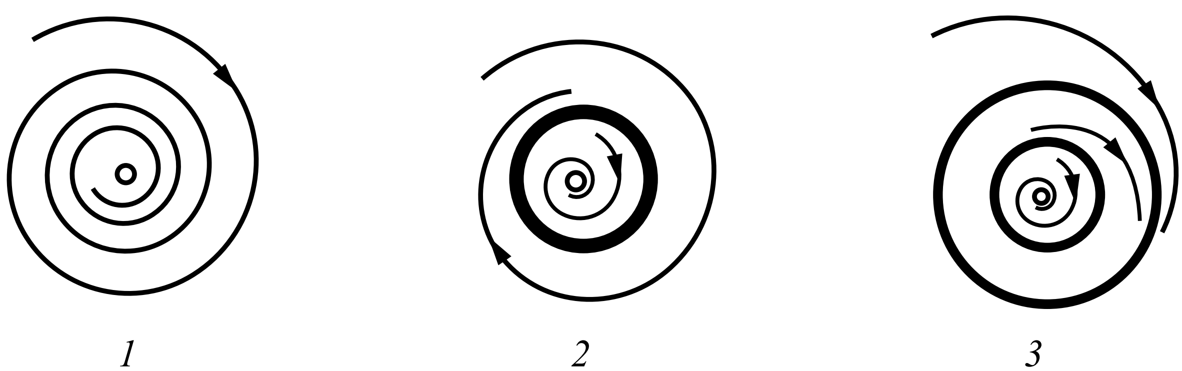


**Figure 3.** Multiple limit cycle bifurcation

# Local Results

- **N. N. Bautin** (1952): $H_o(2) = 3$
  **H. Żołądek** (1995): $H_o(3) \geq 11$

- **F. Dumortier, R. Roussarie,**
  **C. Rousseau** (1994):
  classification and cyclicity
  of quadratic separatrix cycles

- **L. M. Perko** (1995):
  bifurcations of multiple limit cycles

# Global Results

- **Shi Sonling** (1979);
  **Chen Lansun, Wang Mingshu** (1979):
  $H(2) \geq 4$ and $(3:1)$ - distribution

- **R. Bamón** (1986):
  $H(2) < +\infty$

- **Yu. S. Il'yashenko** (1987);
  **J. Écalle, J. Martinet, R. Moussu,**
  **J.-P. Ramis** (1987):
  $H(n) < +\infty$

## Fundamental Ideas

- **N. P. Erugin** (1950):
  qualitative investigation on the whole

- **G. F. D. Duff** (1953):
  field rotation parameters

- **A. Wintner** (1931);
  **L. M. Perko** (1990):
  termination principle of multiple limit cycles

# Quadratic Canonical Systems

**Theorem.** *Any quadratic system with limit cycles can be reduced to one of the canonical forms:*

$$\begin{aligned}\dot{x} &= -y\,(1 + x + \alpha\, y),\\ \dot{y} &= x + (\lambda + \beta + \gamma)\, y + a\, x^2 \\ &\quad +(\alpha + \beta + \gamma)\, xy + c\,\gamma\, y^2\end{aligned} \qquad (C_1)$$

*or*

$$\begin{aligned}\dot{x} &= -y\,(1 + \nu\, y), \quad \nu = 0; 1,\\ \dot{y} &= x + (\lambda + \beta + \gamma)\, y + a\, x^2 \\ &\quad +(\beta + \gamma)\, xy + c\,\gamma\, y^2.\end{aligned} \qquad (C_2)$$

---

Another pair of canonical forms:

$$\begin{aligned}\dot{x} &= -y\,(1 + x) + \alpha\, Q(x, y),\\ \dot{y} &= x + \lambda\, y + a\, x^2 + \beta\, y\,(1 + x) + c\, y^2 \\ &\equiv Q(x, y)\end{aligned} \qquad (C_3)$$

*or*

$$\dot{x} = -y + \nu\, y^2, \quad \dot{y} = Q(x, y), \quad \nu = 0; 1. \qquad (C_4)$$

# An Example of at Least Four Limit Cycles

A quadratic canonical system with two field rotation parameters:

$$\begin{aligned} \dot{x} &= P(x,y) + \alpha\, Q(x,y), \\ \dot{y} &= Q(x,y) - \alpha\, P(x,y), \end{aligned} \qquad (C_5)$$

where

$$P(x,y) = -y + b_{11}\, xy + (b_{02} - \gamma)\, y^2,$$
$$Q(x,y) = x - x^2 + \gamma\, xy + a_{02} y^2.$$

**Example.** $b_{02}^2 - 4(b_{11} - 1)a_{02} < 0,\ b_{02} > 0,$

$$g_3^0 > 0,\ g_5 < 0$$

for $a_{02} = 10,\ b_{11} = 14,\ b_{02} = 3,\ \alpha = 10^{-6},$ where $g_3,\ g_5$ are respectively the first and second focus quantities of the focus $O(0,0)$ of system $(C_5)$ for $\alpha = 0;\ \gamma = 0 :\ g_3^0 = g_3(0).$

**Theorem.** *A quadratic systems has at least four limit cycles in* $(3:1)$ *- distribution.*

# Four Limit Cycles

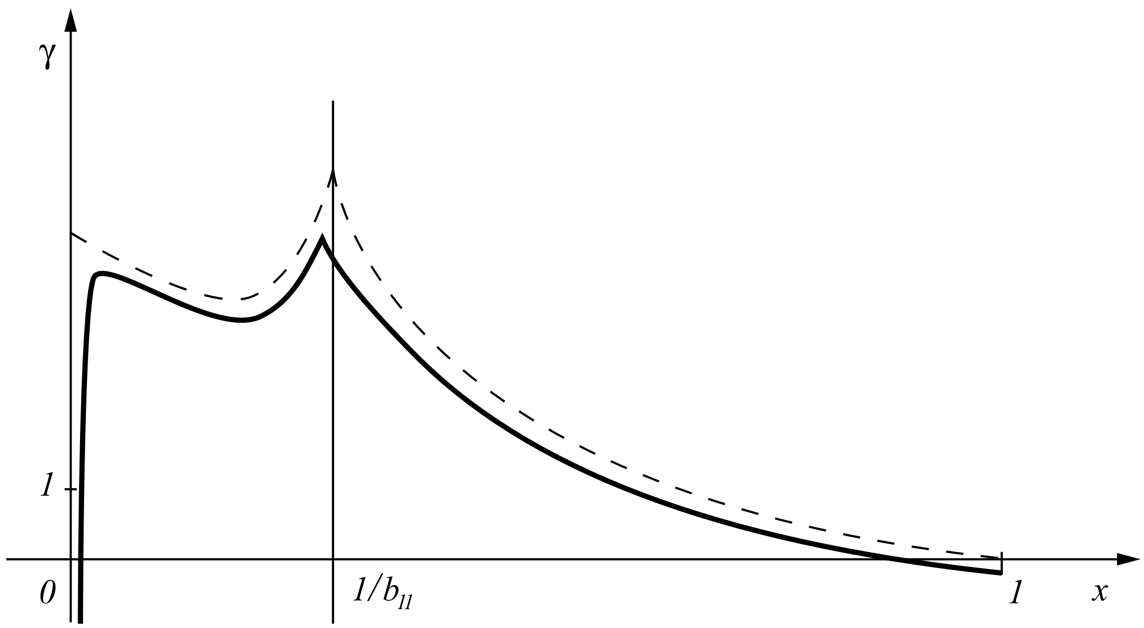


**Figure 10.** Function of limit cycles

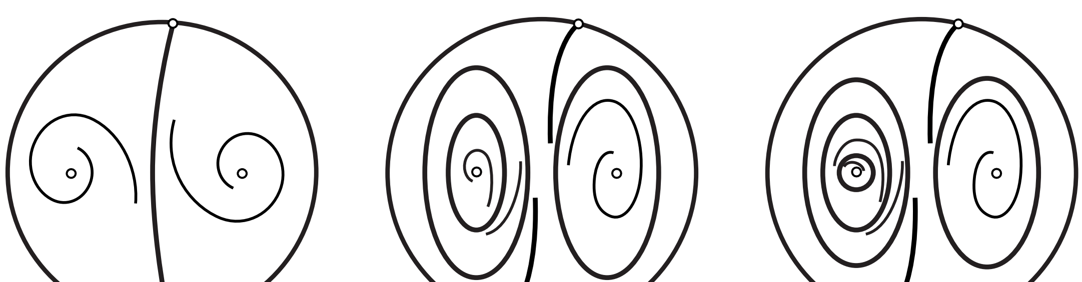


**Figure 11.** Four limit cycles

# Classification of Separatrix Cycles

The classification is carried out in the systems $(C_3)$ and $(C_4)$ according to the number and character of finite singularities:

- one saddle and three antisaddles
- three saddles and one antisaddle
- two saddles and two antisaddles
- one simple saddle and one antisaddle
- two simple antisaddles
- degenerate cases

---

Control of singular points at infinity is carried out with the help of a bundle of cubic curves

$$f(u) = -\alpha c u^3 - (\alpha\beta - (c+1))u^2 - (\alpha a - \beta)u + a,$$

$$u = y/x.$$

It is used the corresponding cases of a center in the origin with $x$-axial symmetry of the vector field (when $\alpha = \beta = \lambda = 0$) and successive variation of the parameters $\lambda$, $\beta$, and $\alpha$.

# Infinite Singularities

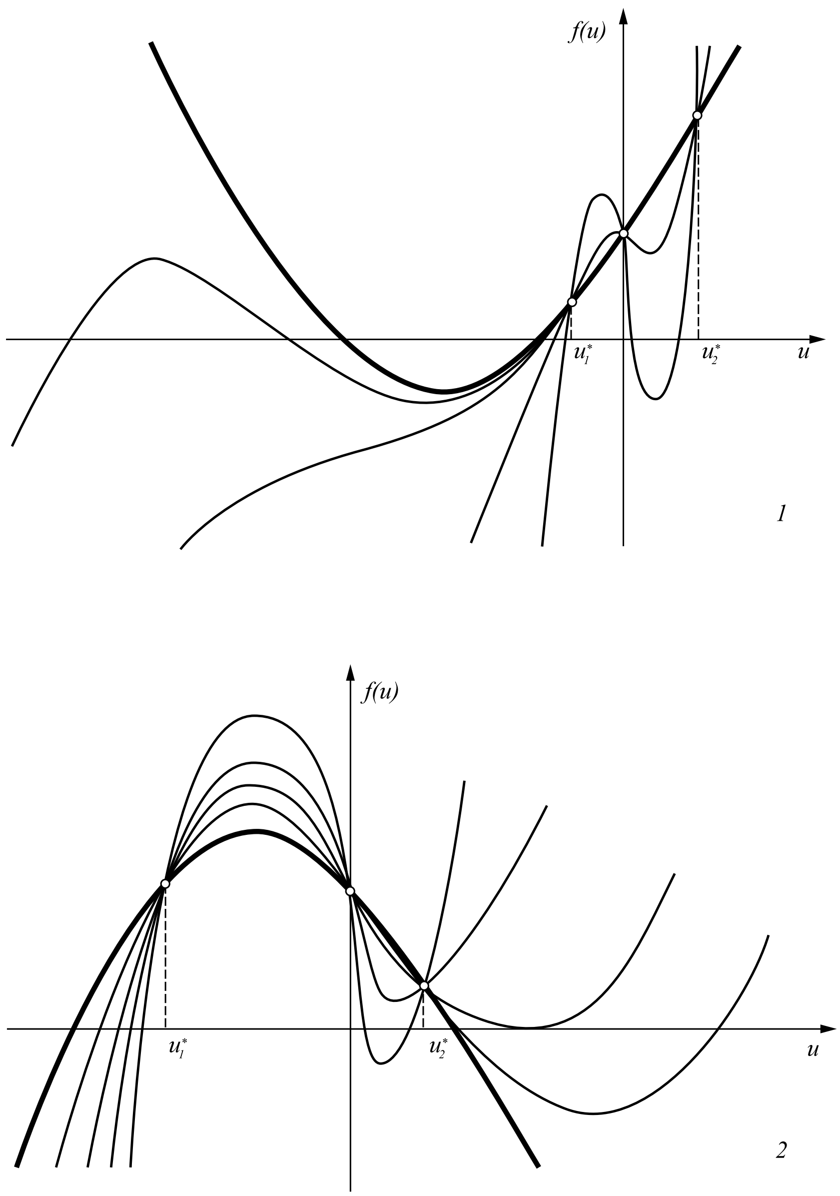


**Figure 12.** Bundles of cubic curves for infinite singularities

# Classification of Separatrix Cycles (Continuation)

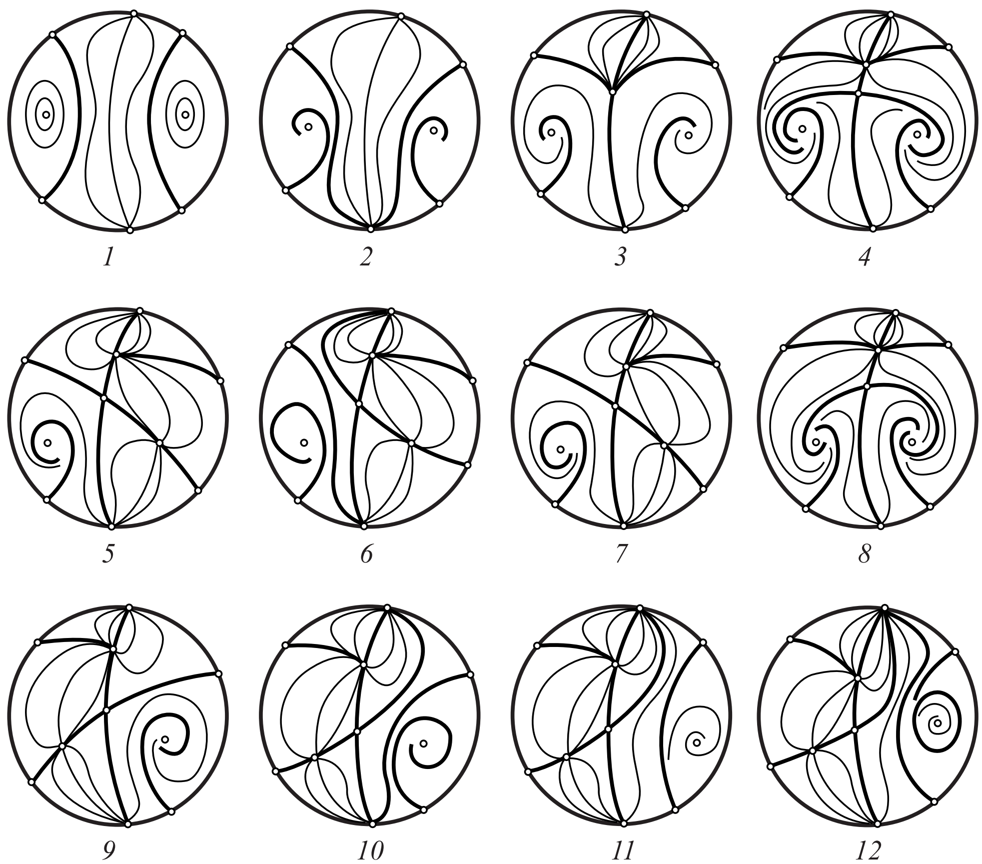


**Figure 13.** Carrying out the separatrix cycle classification in the case when $0 < a < 1$, $c < -1$, $\lambda > 0$, $\beta < 0$, $\alpha = 0$

# Classification (Continuation)

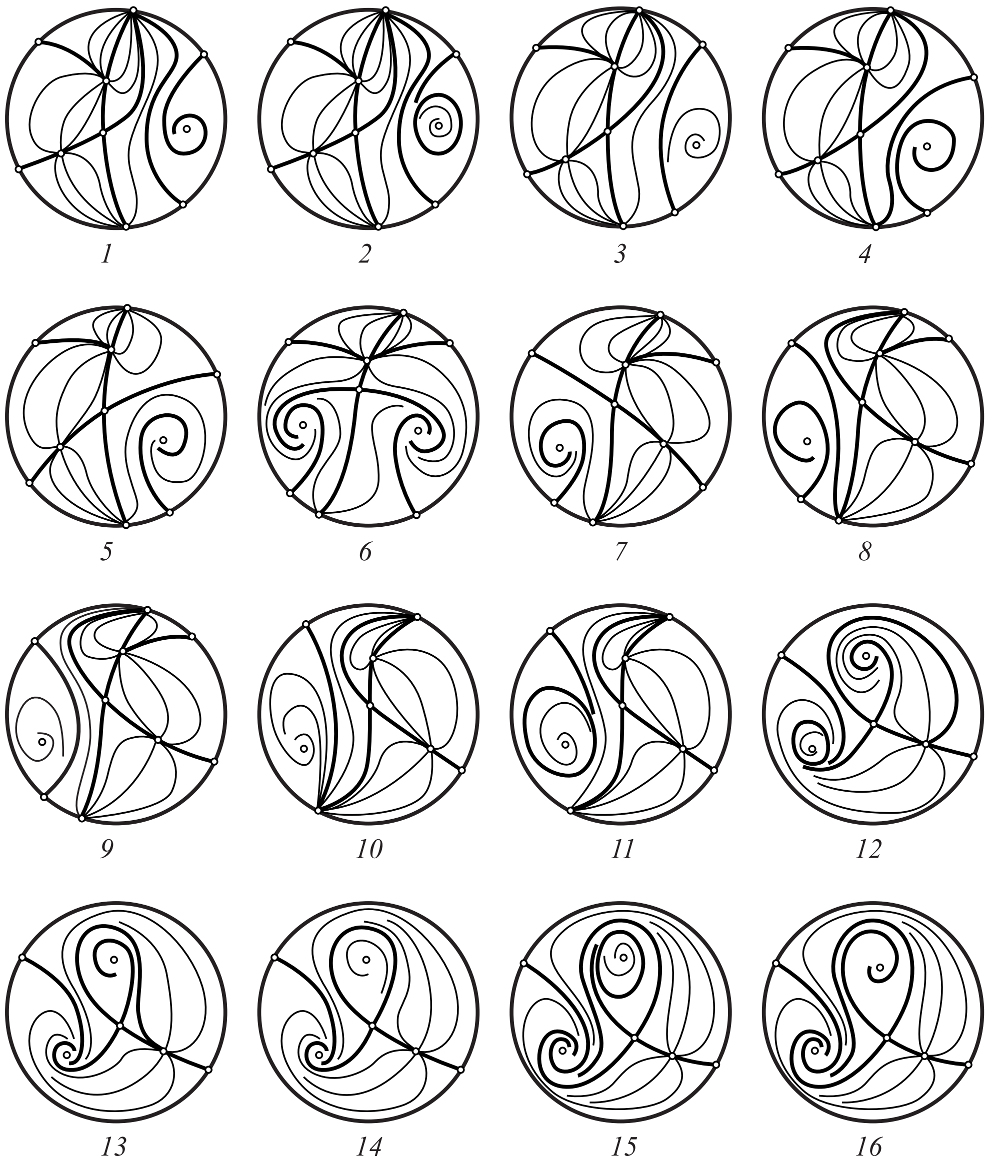


**Figure 14.** The case: $0 < a < 1$, $c < -1$, $\lambda > 0$, $\beta < 0$, $\alpha > 0$

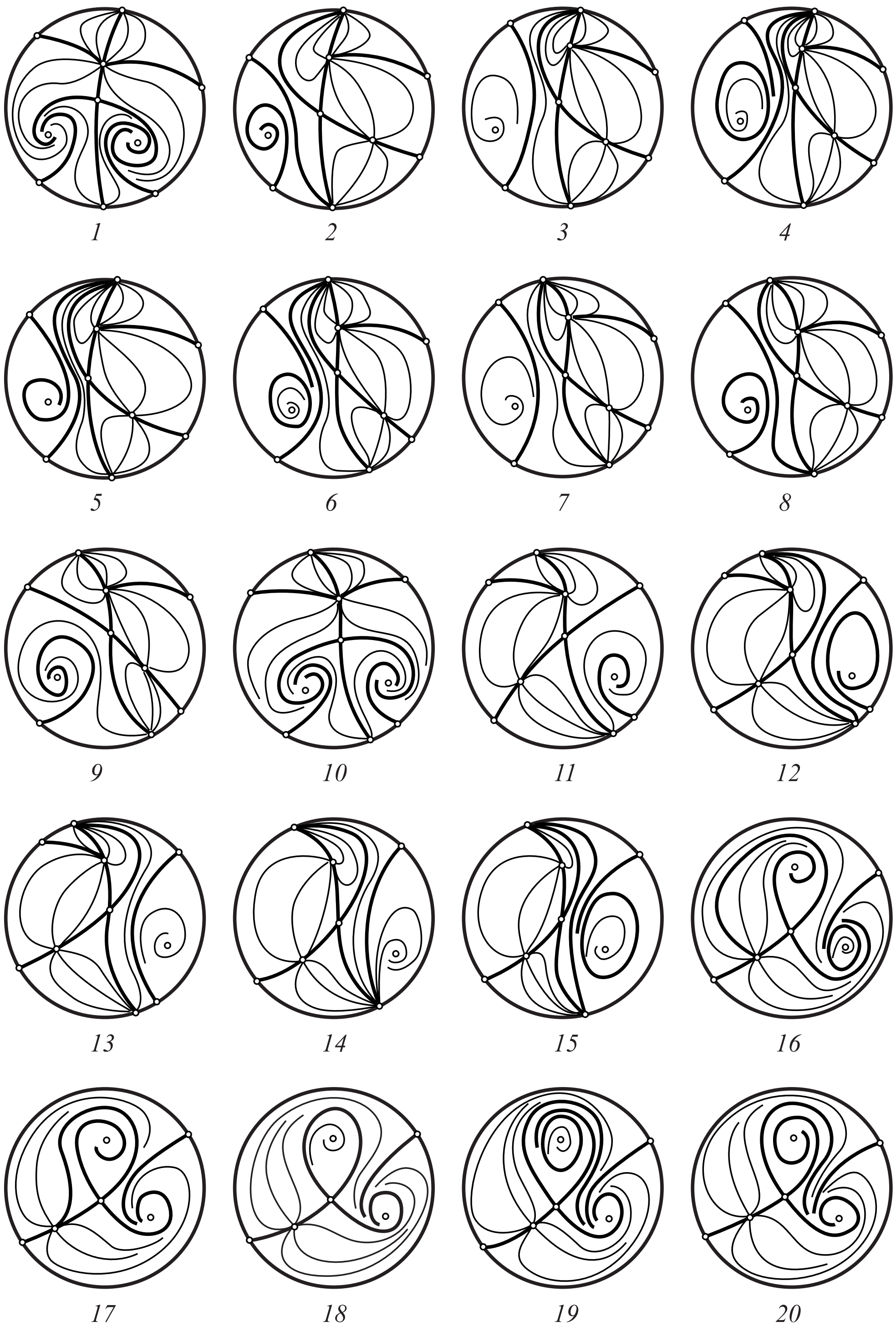


**Figure 15.** The case: $0 < a < 1$, $c < -1$, $\lambda > 0$, $\beta \geq 0$, $\alpha < 0$

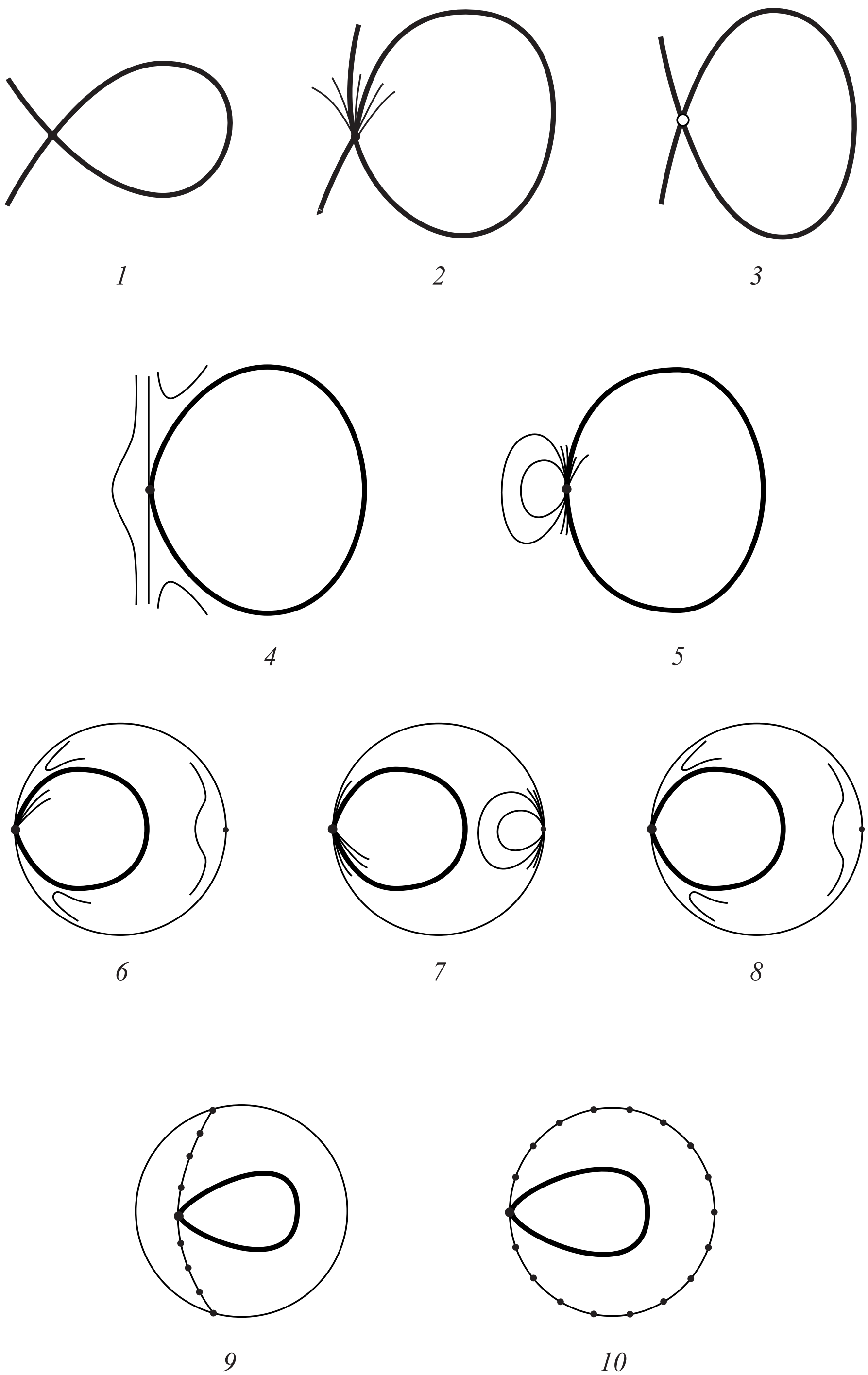


**Figure 16.** Loops

# Digons

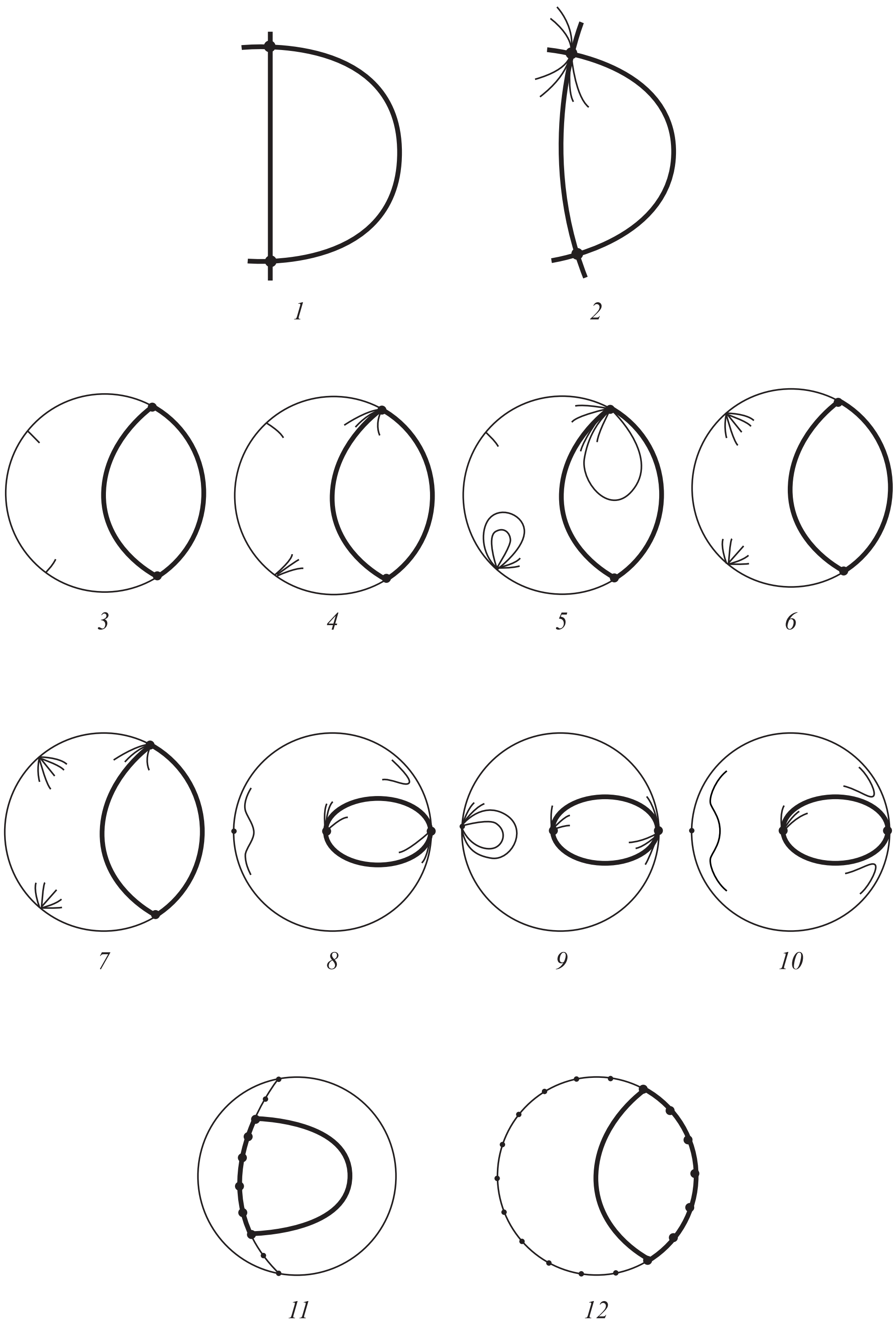


**Figure 17.** Digons

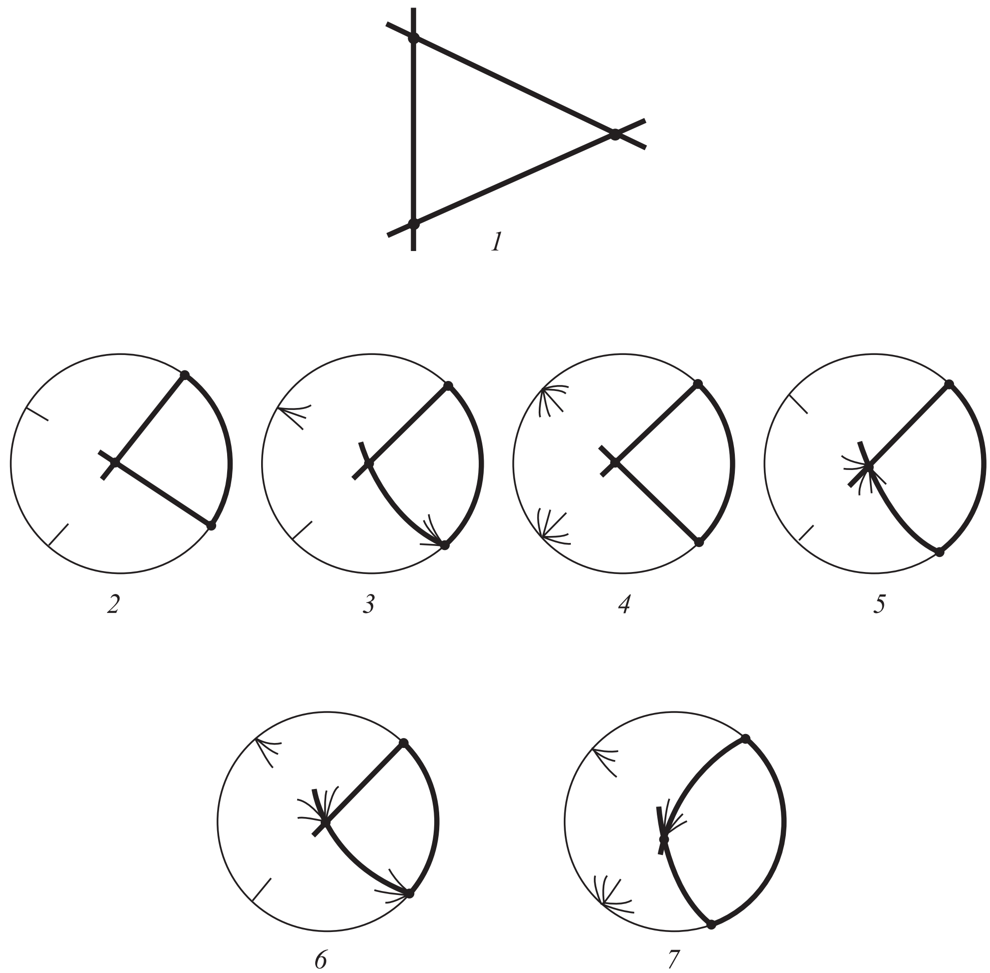


Figure 18. Triangles

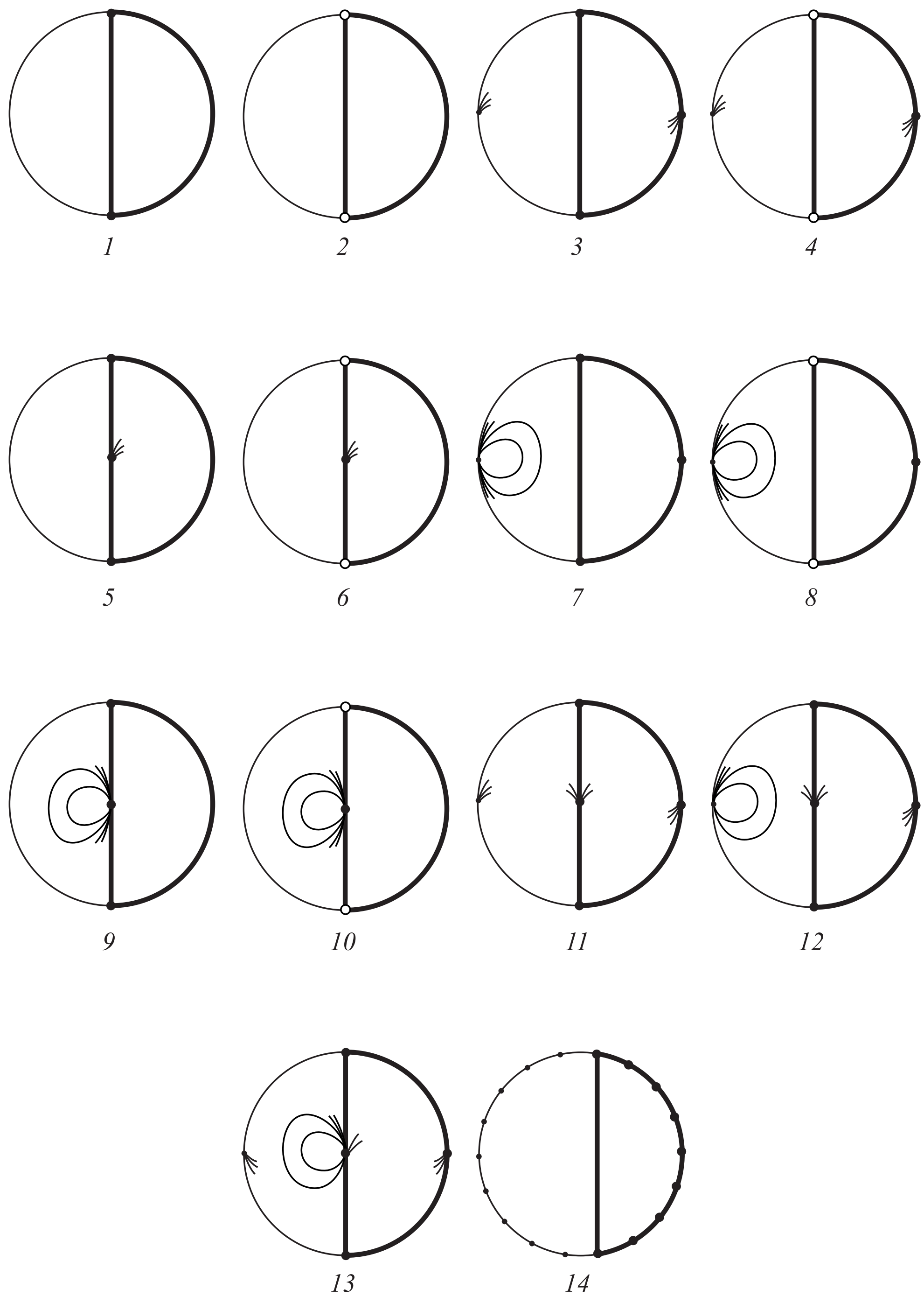


**Figure 19.** Poincaré hemi-cycles

# Multiple Limit Cycles

A two-dimensional $n$-parameter polynomial system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \qquad (M)$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ (polynomial).

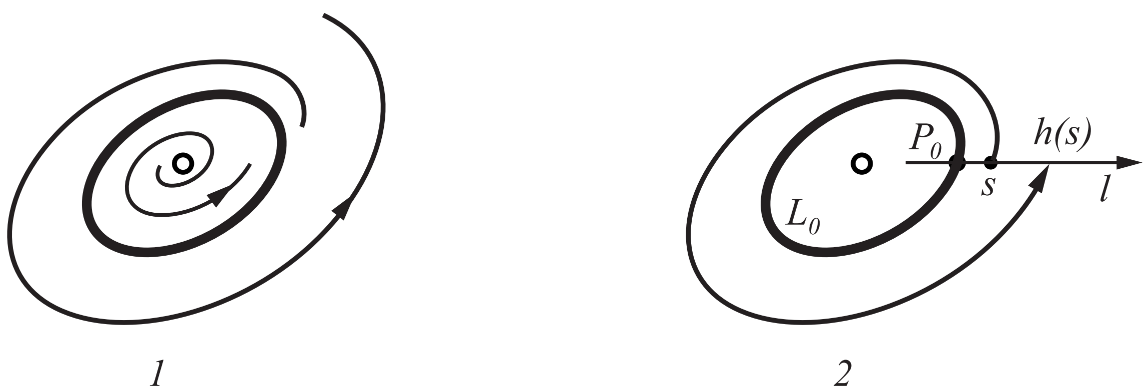


**Figure 20.** Poincaré return map

- $L_o : \boldsymbol{x} = \boldsymbol{\varphi}_o(t)$ is a *limit cycle* at $\boldsymbol{\mu} = \boldsymbol{\mu}_o \in \mathbf{R}^n$
- $h(s, \boldsymbol{\mu})$ is the *Poincaré map*, where

  $l$ is the *normal* to $L_o$ at $\boldsymbol{p}_o = \boldsymbol{\varphi}_o(0)$;

  $s$ is the *coordinate* along $l$
- $d(s, \boldsymbol{\mu}) = h(s, \boldsymbol{\mu}) - s$ is the *displacement function*

**Definition.** A limit cycle $L_o$ of the system $(M)$ is a *limit cycle of multiplicity $m$* iff

$$d(0, \boldsymbol{\mu}_o) = d_s(0, \boldsymbol{\mu}_o) = \ldots = d_s^{(m-1)}(0, \boldsymbol{\mu}_o) = 0,$$

$$d_s^{(m)}(0, \boldsymbol{\mu}_o) \neq 0.$$

# Derivatives of Displacement Function

First partial derivatives along the limit cycle $\boldsymbol{\varphi}_o(t)$:

$$d_s(0, \boldsymbol{\mu}_o) \;=\; \exp \int_0^{T_o} \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{\varphi}_o(t), \boldsymbol{\mu}_o)\, \mathrm{d}t - 1;$$

$$d_{\mu_j}(0, \boldsymbol{\mu}_o) \;=\; \frac{-\omega_o}{\|\boldsymbol{f}(\boldsymbol{\varphi}_o(0), \boldsymbol{\mu}_o)\|}$$

$$\times \int_0^{T_o} \exp\left(-\int_0^t \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{\varphi}_o(\tau), \boldsymbol{\mu}_o)\, \mathrm{d}\tau\right)$$

$$\times \boldsymbol{f} \wedge \boldsymbol{f}_{\mu_j}(\boldsymbol{\varphi}_o(t), \boldsymbol{\mu}_o)\, \mathrm{d}t,$$

where $j = 1, \ldots, n$; $\omega_o = \pm 1$ according to whether $L_o$ is positively or negatively oriented, respectively; and the wedge product of two vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbf{R}^2$ is defined as

$$\boldsymbol{x} \wedge \boldsymbol{y} = x_1\, y_2 - x_2\, y_1.$$

**Remark.** Similar formulas for $d_{ss}(0, \boldsymbol{\mu}_o)$ and $d_{s\mu_j}(0, \boldsymbol{\mu}_o)$ can be derived in terms of integrals of the vector field $\boldsymbol{f}$ and its first and second partial derivatives along $\boldsymbol{\varphi}_o(t)$.

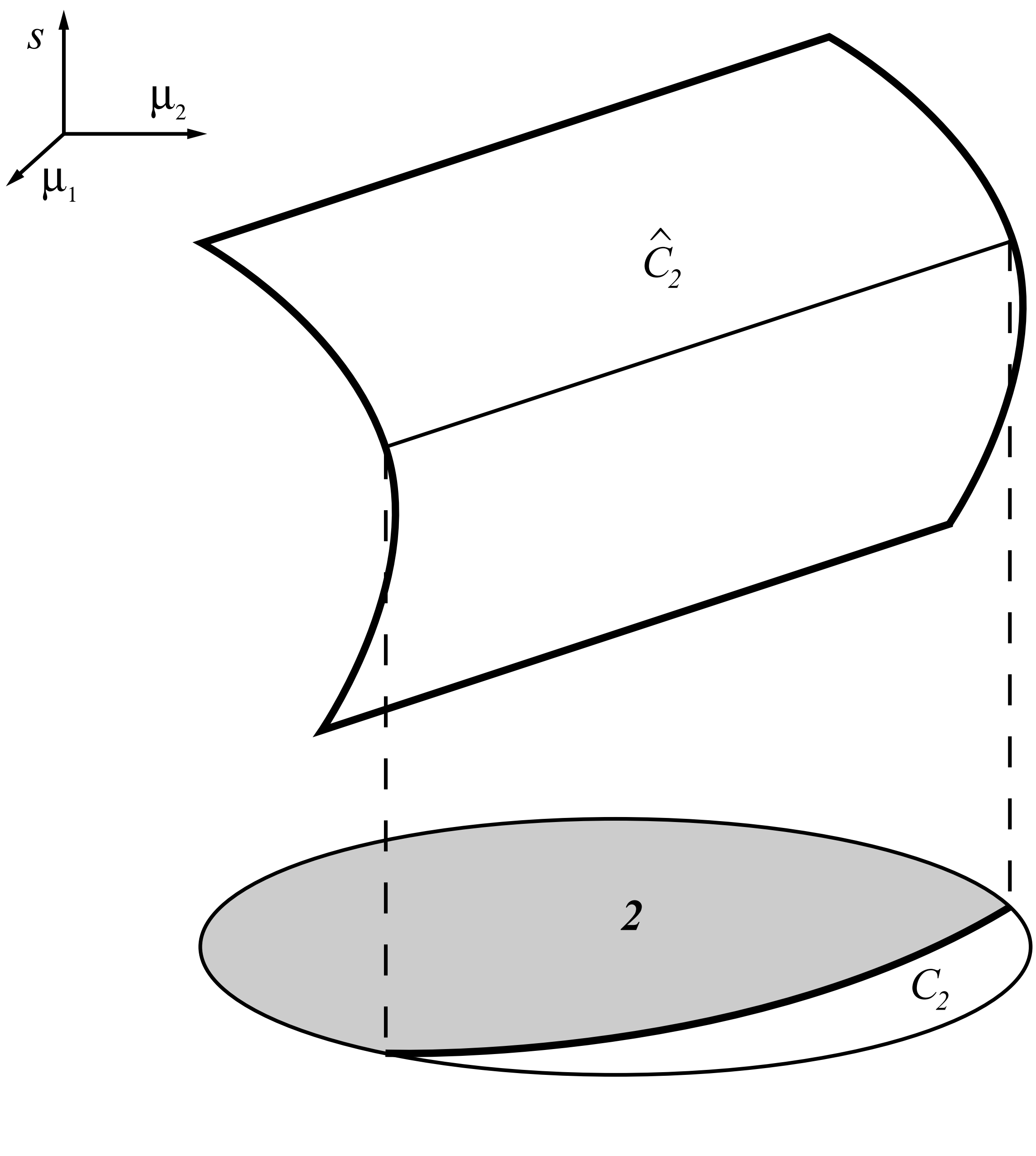


**Figure 21.** Fold bifurcation surface

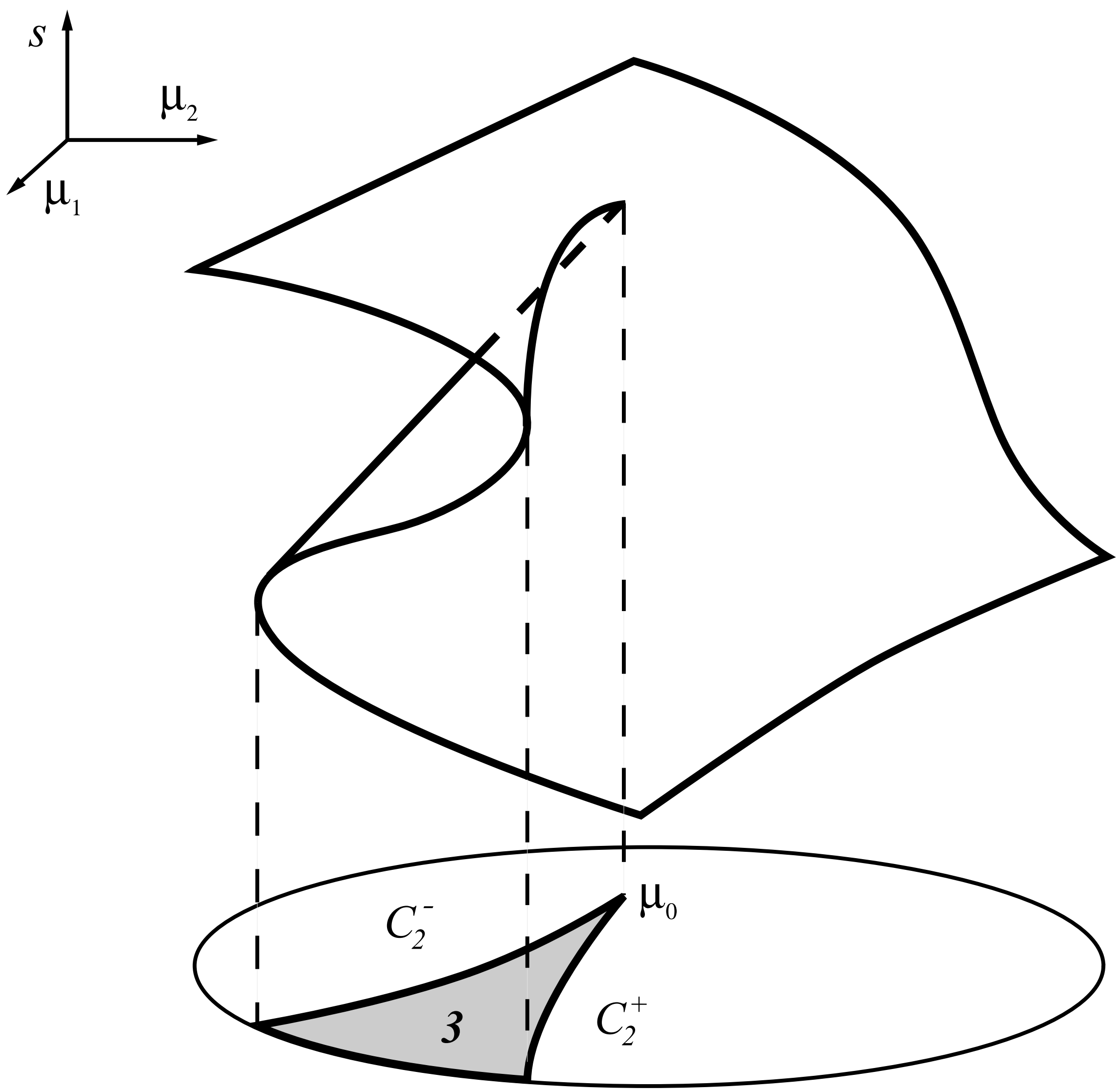


**Figure 22.** Cusp bifurcation surface

# Swallow-Tail

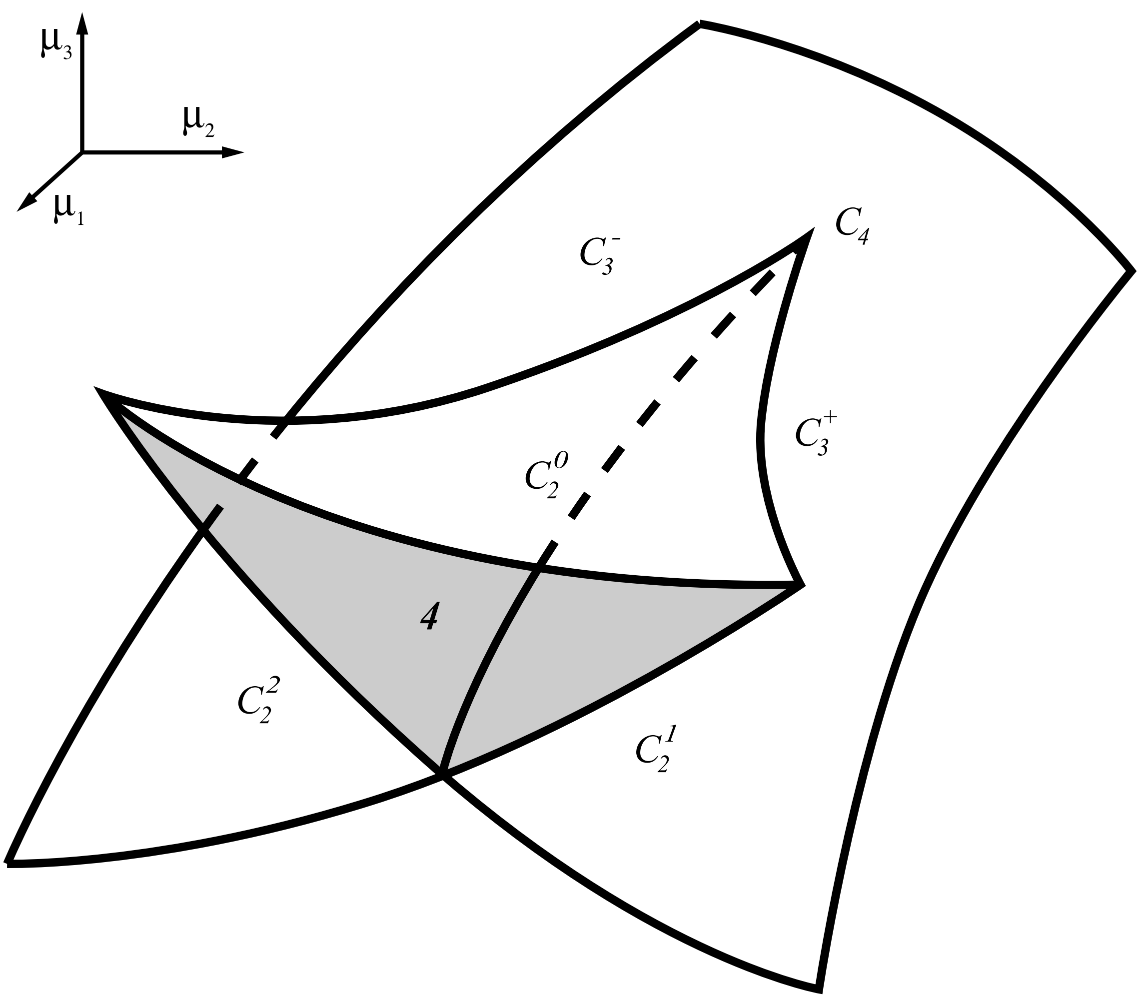


**Figure 23.** Swallow-tail bifurcation surface

# A Curve of Multiple Limit Cycles

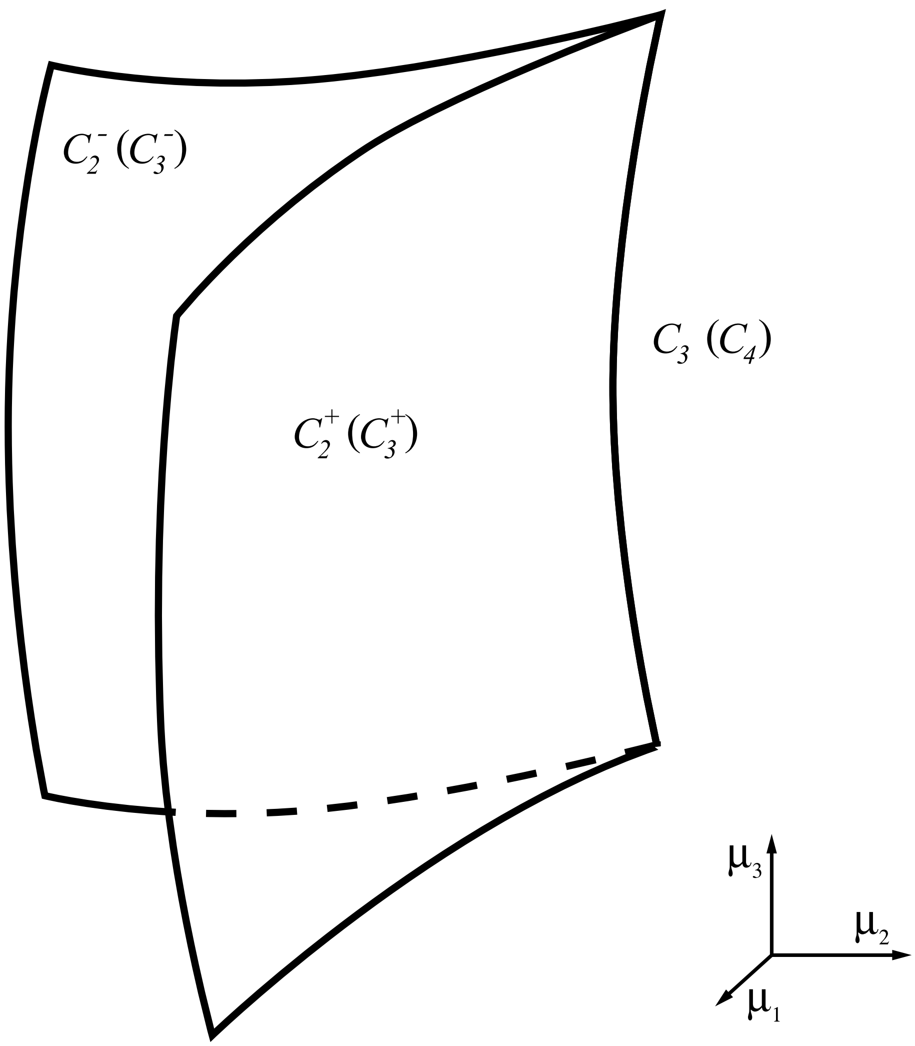


**Figure 24.** **A curve (one-parameter family) of multiple limit cycles**

**Remark.** For the case when $n = m$ (i. e., when the number of parameters is equal to the multiplicity of limit cycles) we obtain a local curve (one-parameter family) of multiplicity-$m$ limit cycles of $(M)$ $(n \geq m \geq 2)$.

# Wintner – Perko Termination Principle

**Theorem (Wintner – Perko).** *Any one-parameter family of multiplicity-$m$ limit cycles of the relatively prime polynomial system $(M)$ can be extended in a unique way to a maximal one-parameter family of multiplicity-$m$ limit cycles of $(M)$ which is either open or cyclic.*

*If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of $(M)$, which is typically a fine focus of multiplicity $m$, or on a $($compound$)$ separatrix cycle of $(M)$, which is also typically of multiplicity $m$.*

# Monotonic Families of Limit Cycles

**Theorem (Perko).** *If* $L_o$ *is a multiple limit cycle of* $(M_o)$ *and* $\mu \in$ **R** *is a field rotation parameter of* $(M)$, *then* $L_o$ *belongs to a one-parameter family of limit cycles of* $(M)$; *furthermore*:

1) *if the multiplicity of* $L_o$ *is odd, then the family either expands or contracts monotonically as* $\mu$ *increases through* $\mu_o$;

2) *if the multiplicity of* $L_o$ *is even, then* $L_o$ *bifurcates into a stable and an unstable limit cycle as* $\mu$ *varies from* $\mu_o$ *in one sense and* $L_o$ *disappears as* $\mu$ *varies from* $\mu_o$ *in the opposite sense*; *i. e., there is a fold bifurcation at* $\mu_o$.

## Main Results for Quadratic Systems

**Theorem.** *There exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, a quadratic system cannot have neither a multiplicity-four limit cycle nor four limit cycles around a singular point* (*focus*), *and the maximum multiplicity or the maximum number of limit cycles surrounding a focus is equal to three.*

---

**Theorem (Quadratic Hilbert's 16th Problem).** *The maximum number of limit cycles in a quadratic system is equal to four and their only possible distribution is* $(3:1)$.

# FitzHugh – Nagumo Neuronal Model

The FitzHugh – Nagumo model:

$$\begin{aligned} \dot{V} &= I - W - aV + (a+1)V^2 - V^3, \\ \dot{W} &= \varepsilon(V - \delta\, W), \end{aligned} \qquad (FN)$$

where $V$ is the membrane potential, $W$ is a recovery variable, and $I$ is the magnitude of stimulus current, is a two-dimensional simplification of the classical Hodgkin – Huxley model of the spike dynamics in a biological neuron.

This system can be reduced to the canonical form

$$\begin{aligned} \dot{x} &= (\gamma\delta - 1)\, y + (\gamma - a)\, x + b\, x^2 - c\, x^3, \\ \dot{y} &= x - \delta\, y. \end{aligned} \qquad (M_c)$$

---

**Theorem.** *FitzHugh – Nagumo system $(M_c)$ has at most two limit cycles.*

# Planar Neural Networks

For two input neurons, the learning model of neural networks can be written as a system of two cubic differential equations

$$\dot{x} = ((1-\varepsilon)a + (\varepsilon/2)b)x + ((1-\varepsilon)b + (\varepsilon/2)c)y$$
$$-x(ax^2 + 2bxy + cy^2),$$
$$\dot{y} = ((\varepsilon/2)a + (1-\varepsilon)b)x + ((\varepsilon/2)b + (1-\varepsilon)c)y$$
$$-y(ax^2 + 2bxy + cy^2),$$

where the parameters $\varepsilon$ and $a$, $b$, $c$ represent the probability of synaptic formation and the weight strengths for the synapses attached to the input neurons, respectively (the Oja model).

This system can be reduced to the canonical form

$$\begin{aligned} \dot{x} &= \lambda\, x - y - x(ax^2 + 2bxy + cy^2), \\ \dot{y} &= x + \lambda\, y - y(ax^2 + 2bxy + cy^2). \end{aligned} \qquad (N_c)$$

---

**Theorem.** *System* $(N_c)$ *has at most one limit cycle.*

# Quartic Biomedical and Ecological Model

A quartic predator-prey model:

$$\dot{x} = x\left(1 - \lambda x - \frac{y}{\alpha x^2 + \beta x + 1}\right) \quad \text{(prey)},$$

$$\dot{y} = y\left(-\delta - \mu y + \frac{x}{\alpha x^2 + \beta x + 1}\right) \quad \text{(predator)},$$

where $\alpha \geq 0,\ \delta > 0,\ \lambda > 0,\ \mu \geq 0$ and $\beta > -2\sqrt{\alpha}$ are parameters.

This is a variation on the classical Lotka–Volterra system which can be written in the form

$$\begin{aligned} \dot{x} &= \ \ x((1 - \lambda x)(\alpha x^2 + \beta x + 1) - y) \equiv P, \\ \dot{y} &= -y((\delta + \mu y)(\alpha x^2 + \beta x + 1) - x) \equiv Q. \end{aligned} \qquad (Q)$$

We use also an auxiliary system

$$\dot{x} = P - \gamma Q, \quad \dot{y} = Q + \gamma P, \qquad (Q_\gamma)$$

where $\gamma$ is a field rotation parameter.

---

**Theorem.** *System* $(Q)$ *has at most two limit cycles.*

# Classical Liénard Polynomial System

The classical Liénard polynomial system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \mu_1 y + \mu_2 y^2 + \mu_3 y^3 + \ldots \\ &\quad + \mu_{2k} y^{2k} + \mu_{2k+1} y^{2k+1} \end{aligned} \qquad (L)$$

---

**Theorem.** *System $(L)$ with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \mu_1 y + y^2 + \mu_3 y^3 + \ldots \\ &\quad + y^{2k} + \mu_{2k+1} y^{2k+1}, \end{aligned} \qquad (L_c)$$

*where $\mu_1, \ldots, \mu_{2k+1}$ are field rotation parameters.*

---

**Theorem (Smale's 13th Problem).** *Liénard polynomial system $(L)$ has at most $k$ limit cycles.*

# Arbitrary Polynomial System

An arbitrary polynomial system:

$$\begin{aligned} \dot{x} &= P_n(x, y, \mu_1, \ldots, \mu_k), \\ \dot{y} &= Q_n(x, y, \mu_1, \ldots, \mu_k), \end{aligned} \qquad (P)$$

where $P_n$ and $Q_n$ are polynomials in the real variables $x$, $y$ and not greater than $n$ degree containing $k$ field rotation parameters, $\mu_1, \ldots, \mu_k$, and having an anti-saddle at the origin.

**Theorem.** *Polynomial system $(P)$ containing $k$ field rotation parameters and having a singular point of the center type at the origin for the zero values of these parameters can have at most $k-1$ limit cycles surrounding the origin.*

## Piecewise Linear Dynamical Systems

A Liénard-type dynamical system:

$$\dot{x} = y - \varphi(x), \quad \dot{y} = \beta - \alpha x - y, \qquad (PL)$$
$$\alpha > 0, \ \beta > 0,$$

where $\varphi(x)$ is a piecewise linear function containing $k$ dropping sections and approximating some continuous nonlinear function.

Suppose that the ascending sections of $(PL)$ have an inclination $k_1 > 0$ and the descending (dropping) sections have an inclination $k_2 < 0$. Then the phase plane of $(PL)$ can be divided onto $2k + 1$ parts in every of which $(PL)$ is a linear system: the ascending sections are in $k+1$ strip regions $(I, III, \ldots, 2K+1)$ and the descending sections are in other $k$ such regions $(II, IV, \ldots, 2K)$. The parameters $k_1$, $k_2$, and also $\alpha$ can be considered as rotation parameters for the sewed vector field of $(PL)$.

---

**Theorem.** *System $(PL)$ with $k$ dropping sections and $2k + 1$ singular points can have at most $k + 2$ limit cycles, $k + 1$ of which surround the foci one by one and the last, $(k + 2)$-th, limit cycle surrounds all of the singular points of $(PL)$.*

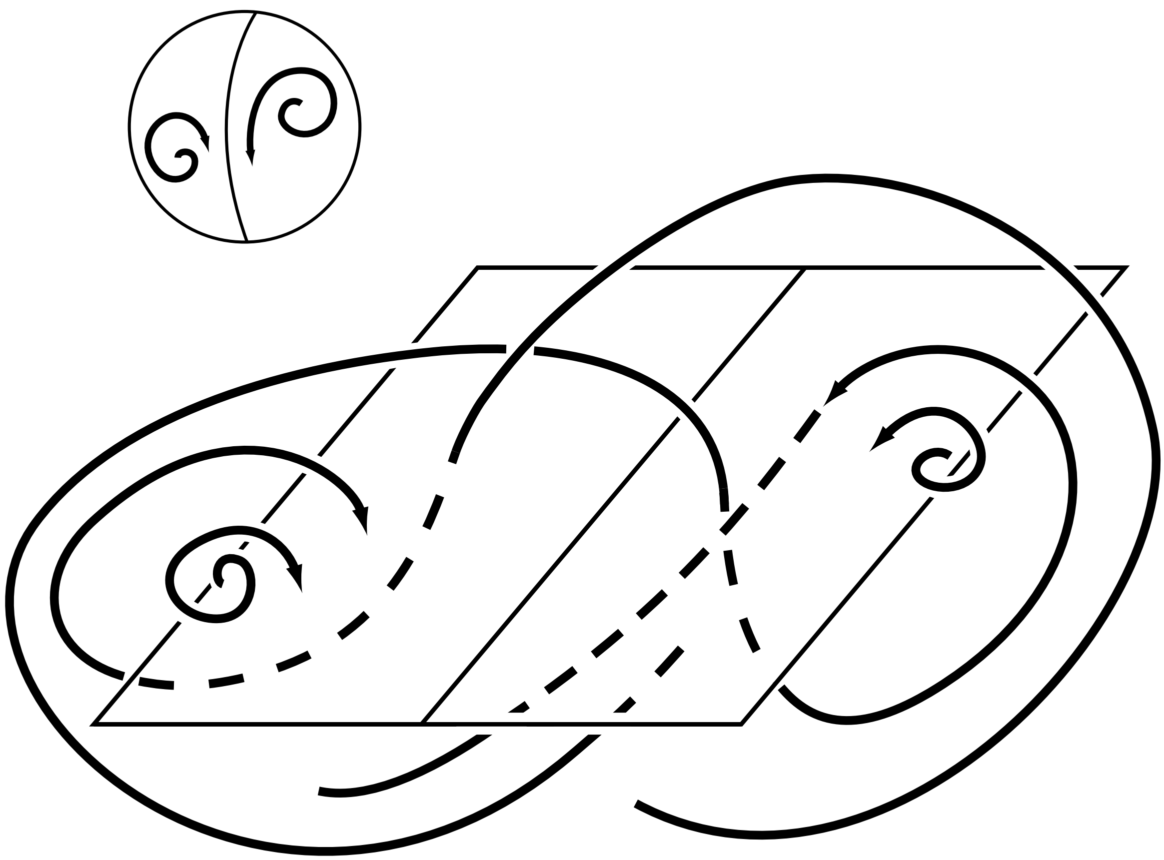

Figure 25. Bifurcation of a strange attractor

# Publications


- F.Botelho and V.A.Gaiko, Global analysis of planar neural networks, *Nonlinear Anal.* **64** (2006), 1002–1011.
- V.A.Gaiko, Limit cycles of quadratic systems, *Nonlinear Anal.* **69** (2008), 2150–2157.
- V.A.Gaiko, Limit cycles of Liénard-type dynamical systems, *Cubo* **10** (2008), 115–132.
- V.A.Gaiko, A quadratic system with two parallel straight-line-isoclines, *Nonlinear Anal.* **71** (2009), 5860–5865.
- V.A.Gaiko and W.T. van Horssen, A piecewise linear dynamical system with two dropping sections, *Int. J. Bifurcation Chaos* **19** (2009), 1367–1372.
- V.A.Gaiko and W.T. van Horssen, Global analysis of a piecewise linear Liénard-type dynamical system, *Int. J. Dyn. Syst. Differ. Equ.* **2** (2009), 115–128.
- H.W.Broer and V.A.Gaiko, Global qualitative analysis of a quartic ecological model, *Nonlinear Anal.* **72** (2010), 628–634.

## Recent Publications


- V.A.Gaiko, Multiple limit cycle bifurcations of the FitzHugh–Nagumo neuronal model, *Nonlinear Anal.* **74** (2011), 7532–7542.
- V.A.Gaiko, On limit cycles surrounding a singular point, *Differ. Equ. Dyn. Syst.* **20** (2012), 329–337.
- V.A.Gaiko, The applied geometry of a general Liénard polynomial system, *Appl. Math. Letters* **25** (2012), 2327–2331.
- V.A.Gaiko, Limit cycle bifurcations of a general Liénard system with polynomial restoring and damping functions, *Int. J. Dyn. Syst. Differ. Equ.* **4** (2012), 242–254.
- V.A.Gaiko, Chaos transition in the Lorenz system, *Herald Odesa Nation. Univ. Math. Mech.* **18** (2013), 51–58.
- V.A.Gaiko, Limit cycle bifurcations of a special Liénard polynomial system, *Adv. Dyn. Syst. Appl.* **9** (2014), 109–123.
- V.A.Gaiko, Global bifurcation analysis of the Lorenz system, *J. Nonlin. Sci. Appl.* **7** (2014), 429–434.