Dynamics of the Thue-Morse system of difference equations

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Introduction

As an initial motivation, we try to understand the two dimension discrete dynamics, that is, given the dynamical system (X, F), where $X \subset \mathbb{R}^2$ and F a continuous map of X into itself, what is the behaviour of the sequences $\mathrm{Orb}_F(P) = \{P, F(P), F^2(P), ...\}$ for every $P \in X$ and try to evaluate its complexity.

Interesting general problems are

1. $$F(x, y) = (p_1(x, y), p_2(x, y))$$
   where p_1, p_2 are quadratic polynomials
2. Lotka-Volterra models
   $$F(x, y) = (x(ax + by + c), y(Ax + By + C))$$

Introduction

Systems obtained by unfolding non-linear difference equations of second order $x_{n+2} = f(x_n, x_{n+1})$

1. $$F_{a,b}(x, y) = (y, ay + b - x^2)$$

   Whitley-Holmes's model

2. $$f_{a,b}(x, y) = (y, 1 + ax - by^2)$$

   Hénon model

Of particular interest are two dimensional systems obtained dealing with sequences of two symbols $\{0, 1\}$ using the procedure of *substitution*: Thue-Morse, Fibonacci, Shapiro and their generalizations. We will concentrate in the Thue-Morse case.

The Thue-Morse Sequence

The Thue-Morse sequence

$$\mathbf{t} = (t_n)_{n \geq 0} = 0110100110010...$$

is an ubiquitous mathematical object.
It comes up in algebra, number theory, combinatorics, topology and other areas. Was introduced by E.Prouhet in 1851 related to a problem of number theory and rescue in 1906 by the Norwegian mathematician Axel Thue as an example of an aperiodic recursively defined sequence of numbers. Finally in the twenties of the last century, M.Morse proved that the sequence is *overlap-free*.
It has different but equivalent definitions.

The Thue-Morse Sequence

Define a sequence of words (a finite sequence of symbols) of 0's and 1's by the following *substitution* rule:

$$X_0 = 0$$
$$X_{n+1} = X_n \bar{X}_n$$

where $\bar{X}$ means change all the 0's in X into 1's and vice versa.

For example, we find

$$X_0 = 0$$
$$X_1 = 01$$
$$X_2 = 0110$$
$$X_3 = 01101001$$
$$X_4 = 0110100110010110$$
$$\cdots$$

The Thue-Morse Sequence

Then

$$\lim_{n\to\infty} X_n = \mathbf{t}$$

The Thue-Morse Sequence

Another way of construction

Representing all non-negative numbers in binary starting by zero

0 1 10 11 100 101 110 111......

The Thue-Morse Sequence.- Words repetitions

An *overlap* is a word of the form $aXaXa$ where a is a single symbol and X ia a word. Examples in English include

alfalfa

entente

A word is *overlap-free* if it contains no word that is an overlap
One good example of overlap-free is the Thue-Morse infinite sequence.

The Thue-Morse sequence

Given the shift space of two symbols $((\Sigma^2, d), \sigma))$, the Thue-Morse sequence $\mathbf{t}$ is a uniform recurrent point of the shift space. The proof was done by Furstenberg proving thet all the words contained in $\mathbf{t}$ are *syndetic*.

The Thue-Morse system of difference equations

In a paper of 1992, Y.Avishai and D.Berend, considered a one dimensional array of N- δ-function potentials

$$V_n(x) = v \sum_{i=1}^{n} \delta(x - x_i)$$

where $v > 0$
and $\mathbf{x} = (x_n)_{n\geq 1}$
is an infinite real sequence whose difference sequence $y_n = x_{n+1} - x_n$ which assumes two possible positive values d_1 and d_2. When we take $\mathbf{x} = \mathbf{t}$, then

The Thue-Morse system of difference equations

$$y_n = x_{n+1} - x_n = d_1 \text{ or } d_2$$

depending on if $\psi_n = 0$ or $\psi = 1$ where $\psi_n = [1 + (-1)^{s(n)}]/2$

Then they study the reflection proccess of a plane wave through a one dimensional array of n-δ-functions located in the Thue-Morse chain with the former distances d_1 and d_2, arriving after application of the Schrödinger equation and some transformations to the following system of difference equations where x_n and y_n for every n are the trace of some matrices associated to the numerical scheme. Finally it is decided if the array behaves as an electrical insulator or conductor.

Thue-Morse quasicrystal

The Thue-Morse sequence is also connected with the description of quasycristals. The following picture is an example of a 1D Thue-Morse quasicrystal where in the two different colour squares there are placed two different type of atoms:

■□□■□■■□□■■□■□□■■□□■□■■□...

The Thue-Morse system of difference equations

The previous problem leads to the following system of difference equations

$$x_{n+1} = x_n(4 - x_n - y_n)$$

$$y_{n+1} = x_n y_n$$

where x_n, y_n are the trace of two matrices associate to the substitution of symbols. The system can be seen as a two dimensional dynamical system given by the pair $(\mathbb{R}^2, T)$ where

$$T(x, y) = (x(4 - x - y), xy)$$

which it is topologically conjugate to the maps:

$$S(x, y) = ((y - 2)^2, xy)$$

or

$$B(x, y) = (xy, (x - 2)^2)$$

For example the conjugation between the two first is given by the map $\Phi(x, y) = (y, 4 - x - y)$.

On the dynamics of The Thue-Morse system, first facts

The most interesting part of the dynamics is concentrated in the interior of the triangle Δ obtained connecting the three points $(0,0),(4,0),(0,4)$. The line Γ connecting $(4,0)$ and $(0,4)$ is given by $x+y=4$. Δ is invariant by H, that is, $H(\Delta)=\Delta$. The segment $\Gamma_1=\{(x,0):x(4-x)\}$ is also invariant $(H(\Gamma_1)=\Gamma_1)$. If $\Gamma_2=\{(0,y):0\leq y\leq 4\}$, then $H(\Gamma)=\Gamma_2$ and $H(\Gamma_2)=\{(0,0)\}$.

The Thue-Morse system, first facts

It is easy to test that all points belonging to IntΔ have no preimages outside it, which means that IntΔ is T-invariant. Also $\partial\Delta$ is also invariant, but there points outside Δ which are preimages of points of $\partial\Delta$.

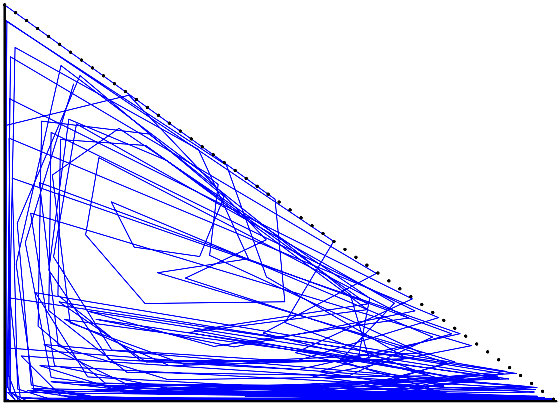


Figure: Orbit of (0.5, 3.2) with 10.000 iterations

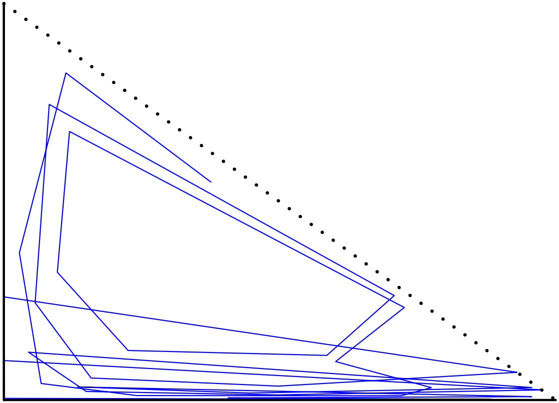


Figure: Orbit of (1.5, 2.2) with 10.000 iterations

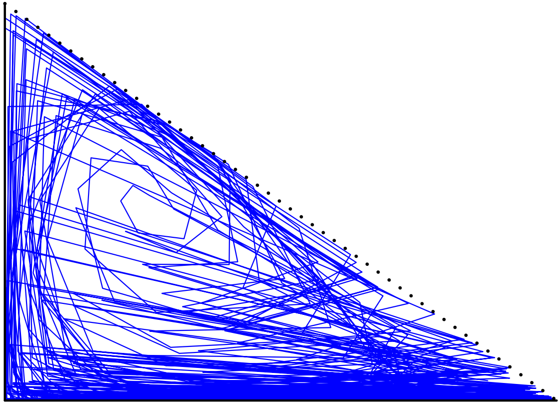


Figure: Orbit of (3.5, 0.2) with 10.000 iterations

The Sharkovskiĭ's program

In 1993 in a conference in Oberwolfach, Sharkovskiĭ proposed to study the two dimensional dynamical system

$$S(x, y) = ((y-2)^2, xy)$$

In particular, to answer to the following questions:

1. Are the periodic points dense in Δ?
2. Is $S|\Delta$ transitive?
3. Is Γ an attractor of Δ in Milnor's sense?
4. Are there points such that $\omega_T(P)$ be unbounded but holding $\omega_T(P) \cap \Gamma \neq \emptyset$

Periodic points

By elementary algebraic computations it is esay to see that inside Δ there is only one fixed point, $(1, 2)$. At the boundary of Δ we have two fixed points, $(0, 0)$ and $(3, 0)$. Outside the triangle we have no fixed point. There are no two-periodic points, neither three periodic points.

By the algebraic method of resultant, we obtain that the interior point.

$$(1 - \sqrt{2}/2, 1 + \sqrt{2}/2)$$

is periodic of period 4 and its orbit is the unique having such a period. By numerical methods we can prove the existence of a unique interior periodic points of period 5 and by direct computation that explicitely

$$(1, (3 + \sqrt{5})/2))$$

is periodic of period six, but it is not unique.

Periodic points

Using a particular symbolic dynamics, P. Malicky has shown that for $n \geq 4$ there is an interior point of T in Δ of period n. It remains open if such points are unique or not.

The key point of Malicky's proof is to prove that given a saddle periodic point P in Γ_2, there exists in the interior of Δ a periodic point having the same itinerary. It is interesting to have a criterium to prove the existence of saddle points in Γ_2. In fact, let $P = [4\sin^2(k\pi/(2^n + (-)1), 0]$, where $n > 0, k$ are integers numbers. If

$$1 \leq k \leq \frac{\sqrt{2}(2^n + (-)1)}{\pi 2^{\sqrt{2n+1/4}}}$$

then P is a saddle fixed point of T^n.

Answering Sharkovskiĭ program

The map $T|\Delta$ is transitive, that is, given two open sets in Δ, U, V there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Instead proving it directly we prove that T is in Δ, *almost topologically exact*. For any non-empty open set U and $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for $n > n_0$ is

Answering Sharkovskiĭ program

1. The set of periodic points is not dense in Δ. The proof uses the following decomposition

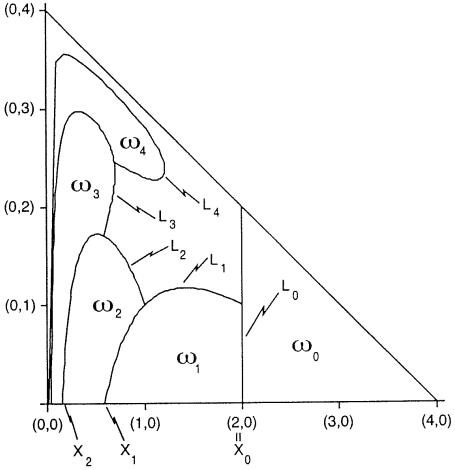
$$T(x,y) = G(\Phi(x,y))$$

   where $\Phi(x,y) = (y, 4-x-y)$ and $G(x,y) = (4-x-y)(x,y)$.
2. The set of periodic points of G is dense in Δ. There are segments inside Δ where Sharkovskiĭ ordering is fullfilled.
3. It is immediate to see that in Δ the maps G and T are not *ergodic* with respect to Lebesgue measure

Answering Sharkovskiĭ program

1. The segments Γ and γ are not *attractors* in the Milnor sense
2. There is no point P whose ω-limit is unbounded and intersecting with Γ

Decomposition of the interior of Δ



Dynamics outside the triangle

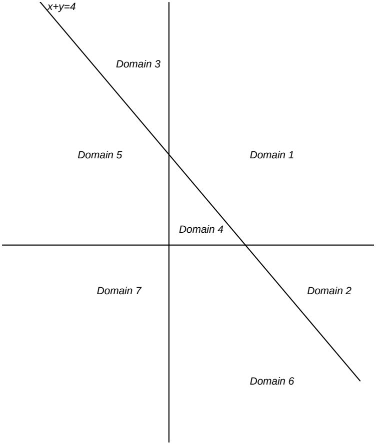


Figure: Partition of the plane

Dynamics outside the triangle

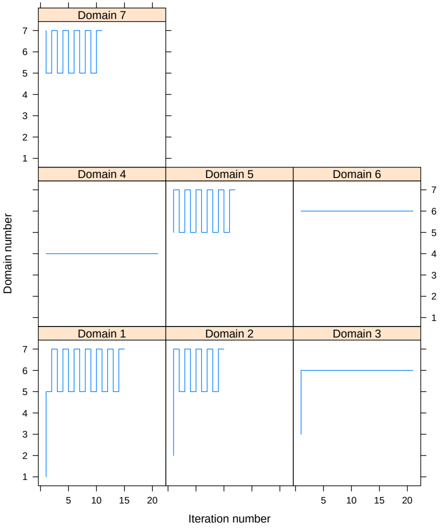


Figure: Partition of the plane

Dynamics outside the triangle

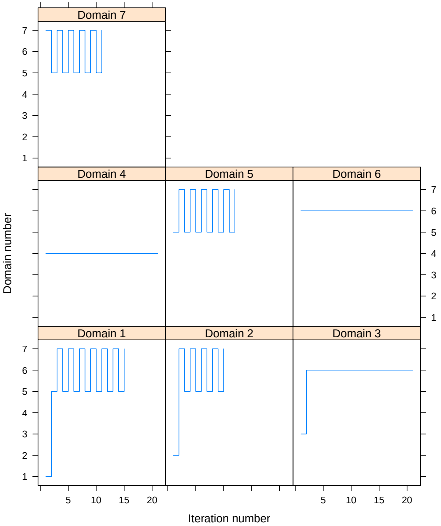


Figure: Partition of the plane

Representation of saddle points

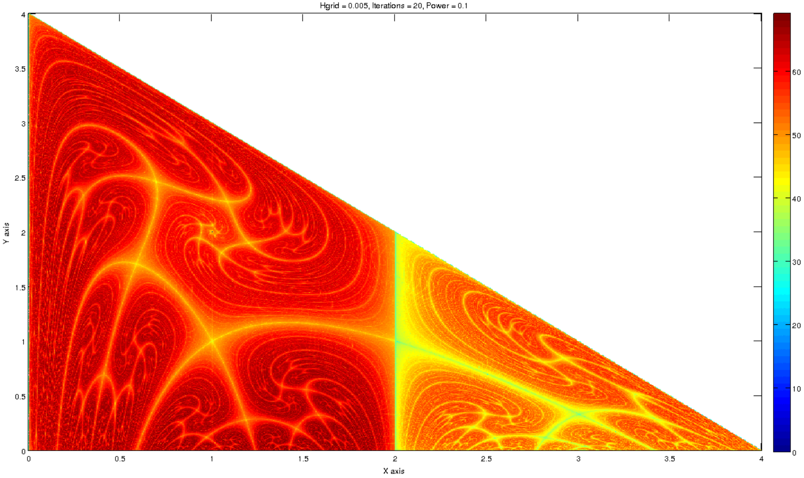


Figure: Triangle

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Fibonacci systems of equations

Y.Avishai and D.Berend in *Transmission through a one-dimensional Fibonacci sequence of δ-function potentials*, repeating what was done in the Thue-Morse case, but using the substitutions rules

$$0 \rightarrow 01$$
$$1 \rightarrow 0$$

we obtain the so called Fibonacci sequence

$$(00101001001....)$$

with it we construct the Fibonacci quasicrystal and then the trace maps associated to the resolution of a partial differential equation, can be solutions of the system of difference equations whose unfolding is

$$F(x, y, z) = (y, z, yx - z)$$

it is open to study the dynamics of such map. In a first step would be of interest detecting, if it is the case, of possible T-invariant sets where the dynamics be concentrate