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One special construction in the spectral theory of C_0-semigroups.

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In joint work with Prof. Dr. Grigory M. Sklyar (Institute of Mathematics, University of Szczecin, Szczecin, Poland & V.N. Karazin Kharkiv National University, Kharkiv, Ukraine) we present the construction of the generator of a C_0-group with the following properties. It has simple eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, which essentially cluster at infinity, i.e. $\lim\limits_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$, and corresponding family of eigenvectors is dense but do not form a Schauder basis. This construction is closely related with the recent results of G.Q. Xu et al. [1] and H. Zwart [2] on Riesz basis property of eigenvectors (eigenspaces) of infinitesimal operators. The discrete Hardy inequality for $p = 2$ plays a key role in our approach.

Let H be a Hilbert space with norm $\|\cdot\|$ and Riesz basis $\{e_n\}_{n=1}^{\infty}$. Consider the operator T defined on H as $Te_n = e_{n+1}$. By $H_1(\{e_n\})$ we then denote the completion of the space $H_1^0(\{e_n\}) = \{x \in H : \|x\|_1 = \|(I-T)\,x\|\}$. It can be shown that

$$H_1(\{e_n\}) = \left\{ x = (\mathbf{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\},$$

where $(\mathbf{f}) \sum\limits_{n=1}^{\infty} c_n e_n$ is a formal series, $\ell_2(\Delta)$ is the space of all sequences whose differences are 2-absolutely summable, and Δ denotes a difference operator. It turns out that $\{e_n\}_{n=1}^{\infty}$ is dense in $H_1(\{e_n\})$ but does not form a Schauder basis in $H_1(\{e_n\})$.

The main result of the work can be formulated as follows. The operator $A : H_1(\{e_n\}) \supset D(A) \mapsto H_1(\{e_n\})$ defined by the formula $Ax = A(\mathbf{f}) \sum\limits_{n=1}^{\infty} c_n e_n = (\mathbf{f}) \sum\limits_{n=1}^{\infty} i \ln n \cdot c_n e_n$, with domain

$$D(A) = \left\{ x = (\mathbf{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \{\ln n \cdot c_n\}_{n=1}^{\infty} \in \ell_2(\Delta) \right\},$$

generates a C_0-group $\{e^{At}\}_{-\infty<t<\infty}$ on $H_1(\{e_n\})$. Moreover, we note that, surprisingly, the constructed C_0-group $\{e^{At}\}_{-\infty<t<\infty}$ has a linear growth when $t \to \pm\infty$.

References

[1] G. Q. Xu, S. P. Yung *The expansion of a semigroup and a Riesz basis criterion.* J. Differential Equations **210** (2005), 1–24.

[2] H. Zwart*Riesz basis for strongly continuous groups.* J. Differential Equations **249** (2010), 2397–2408.